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CONTENTS

# ON THE LOCAL UNIQUENESS PROBLEM FOR PERIODIC SURFACE WAVES OF PERMANENT TYPE IN A CHANNEL OF INFINITE DEPTH[1]

D.H. Hyers and J.A. Ferling

## *Introduction*

We shall be interested in studying wave motions at the surface of a heavy[2] liquid in a channel bounded by parallel vertical walls. we assume that the motion is the same in each vertical plane parallel to the walls, so that the problems considered are two dimensional, and that the flow is incompressible, non-viscous, and irrotational.

There are two characteristic difficulties of such problems. First, the wave profile (surface) is an unknown of the problem so that even the region of flow is unknown in advance. Secondly, one of the conditions to be fulfilled along this unknown or "free" boundary is *non-linear* in character. It is obtained from Bernoulli's law involving the *square* of the velocity, the particle ordinate and the pressure, by stipulating that the pressure is constant along the free boundary (E.g., at an air-water interface the pressure would be atmospheric pressure).

In most discussions of such problems the standard procedure is to linearize the equations by considering only small motions and neglecting second order quantities. One also replaces the (unknown) free boundary by a fixed straight line (equilibrium surface of liquid) in stipulating the linearized boundary condition. The resulting linear problem for a known region constitutes only a first order approximation to the original problem, but is comparatively easy to solve.[2]

One of the simplest types of such wave motion is a traveling wave of "permanent type." This denotes a disturbance which travels down

1. This paper represents results obtained under contract Nonr 228-09 (NR 041/152) with the Office of Naval Research.

2. The adjective heavy signifies that we do not neglect the force of gravity.

3. See e.g., Lamb's *Hydrodynamics*, chapter 9.

the channel at a constant velocity of propagation $c$, and without change of form. In more mathematical terms, this means that the motion is *steady* when viewed from a coordinate system attached to a point of the profile moving with the constant velocity $c$.

In the early part of the century there was considerable discussion as to whether these waves of permanent type actually exist, and for a time Lord Rayleigh himself had doubts about the matter. A number of different approximate methods of solution had been devised, but the first mathematical demonstrations of the existence of exact solutions were given in the early 1920's by Levi-Civita [1] and by Nekrasov[4] for the case of periodic waves in an infinitely deep channel.

Let $\lambda$ denote the wave length, $g$ the acceleration of gravity. The characteristic dimensionless parameter may be taken to be $\gamma = \frac{g\lambda}{2\pi c^2}$ where $c$ is the velocity of propagation as explained above.

Levi-Civita's procedure was first to transform the problem as explained below in section one, making use of the periodicity as well as of the other hypotheses to introduce a new independent (complex) variable $\zeta$ which varies over the unit circle in the $\zeta$-plane. This has the advantage of eliminating the first characteristic difficulty of an unknown domain for the independent variable. At the same time he introduced a new dependent variable $\omega = \tau + i\theta$ related to the velocity. The condition of constant pressure along the free boundary was transformed into the non-linear condition $\frac{\partial\tau}{\partial\varphi} = -\gamma e^{-3\tau}\sin\theta$ to be fulfilled along the circumference $\zeta = e^{i\varphi}$ of the unit circle in the $\zeta$-plane. By assuming an expansion for both $\omega$ and $\delta$ in terms of a certain real parameter $\mu$, he was able to determine the coefficients in both expansions and finally to demonstrate the convergence of these expansions in the neighborhood of $\mu = 0$. Thus the existence of exact solutions was demonstrated locally.

However, it seems that no adequate discussion of uniqueness of solutions of this problem has yet appeared in the literature. This paper represents a first step in this direction, being concerned only with *local* uniqueness.

We use a method explained by O.P.Friedrichs [2] to reformulate the problem as a problem of "bifurcation." (cf. sec. 2.) In this approach, the parameter $\mu$ of Levi-Civita arises naturally, and Levi-Civita's ad hoc assumption of the expansions in powers of $\mu$ is avoided.

As a by-product of our method, we prove that a conjecture of Levi-Civita[5], namely that every solution with $\gamma > 1$ is simply related to a solution with $\delta < 1$, is true at least "locally", i.e. for $\gamma$ in the neighborhood of each integer. Our results are given in section 3.

4. We have not had access to Nekrasov's papers [5], but only to an abstract of them given by Levi-Civita.
5. [1], p. 284.

*1. Levi-Civita's formulation of the problem.*

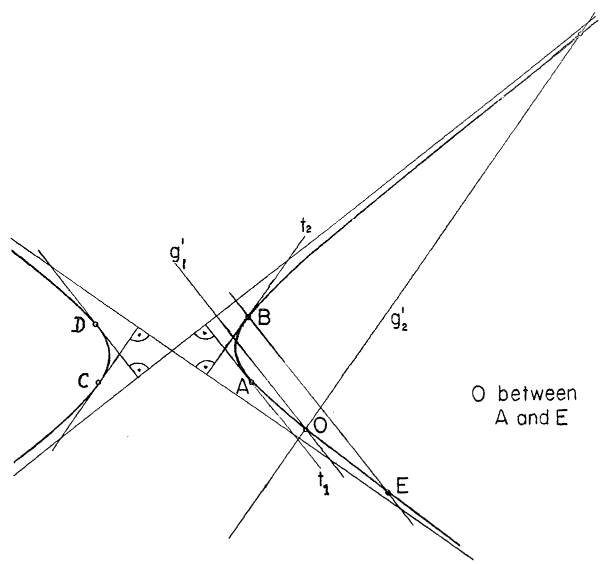


Figure 1.

Let $c$ denote the velocity of propagation of the wave and let $\lambda$ denote the wave-length (assumed independent of the time). The $(x, y)$-coordinate axes are to be fixed to the profile $l$ of the wave, with the origin at a point of $l$ and with the $y$-axis pointing vertically downwards. Our assumptions imply that the flow when viewed from this coordinate system is steady, two dimensional, irrotational, and incompressible, and is periodic with the period $\lambda$.

We denote the components of the particle velocity relative to the coordinate axes by $(u, v)$. The actual velocity (relative to an observer fixed to the earth) is then $(u - c,\ v)$. We shall assume that the actual velocity is zero at infinite depth, so that $u \to c$ and $v \to 0$ as $y \to \infty$. Setting $w = u - iv$, we see that $\overline{w} = u + iv$ is the (relative) velocity vector in the complex $z$-plane, where $z = x + iy$.

The irrotationality of the motion and the incompressibility of the fluid allow us to introduce the velocity potential $\Phi$, the stream function $\Psi$, and the complex potential $\chi = \Phi + i\Psi$, where $\frac{d\chi}{dz} = w$.

The functions $\Phi$ and $\Psi$ being determined only up to additive constants, we may adjust the latter so that $\Phi = \Psi = 0$ at the origin. Thus in particular $\Psi = 0$ on the free boundary $l$ and $\chi(0) = 0$.

The region of flow (including the free boundary $l$) in the $z$-plane maps into the half-plane $\Phi \geq 0$ in the $\chi$-plane (Figure 1).

From the assumption that the function $w(z)$ has the real period $\chi$ it follows easily that $\chi(z+\lambda) = \chi(z) + c\lambda$. Thus if we look upon the complex velocity $w$ as a function of $\chi$, $w$ will have the real period $c\lambda$. For this reason it is convenient to introduce a new in-independent variable $\zeta$ defined by

$$\zeta = e^{\frac{2\pi}{c\lambda} i\chi} = e^{\frac{2\pi}{c\lambda}(-\Psi + i\Phi)}.$$

Considering now the mapping from the $z$-plane to the $\zeta$-plane we see that it is many to one. However two values of $z$ differing by $\lambda$ correspond to the same point $\zeta$ under this mapping, so that $w$ is a single valued function of $\zeta$. Clearly, points of the free boundary $l$ correspond to points of the circle $|\zeta| = 1$ and the "point" $y = +\infty$ corresponds to $\zeta = 0$. If we restrict ourselves to a period strip in the $z$-plane, bounded by the profile $l$ and the equipotential surfaces $\Phi = \frac{1}{2}c\lambda$, we have a simple map onto the unit disc in the $\zeta$-plane (see Figure 1).

The origin in the $z$-plane is mapped into the point $\zeta = 1$, but so far $z = 0$ is an arbitrary point of the profile $l$. For the present, we leave this choice of origin abritrary, but fixed. Later on, the choice of $z = 0$ will be made[6] by requiring that $dw/d\zeta$ shall be real and negative for $\zeta = 0$, which has the effect of specifying one direction in the $\zeta$-plane.

To obtain the boundary condition along the free surface $l$ we observe that the pressure there is constant. Then by Bernoulli's law,

$$\frac{1}{2}(u^2 + v^2) - gy = \text{const. along } l, \tag{1.1}$$

where $g$ denotes the acceleration of gravity.

In order to simplify the problem we introduce a new dependent variable $\omega = \tau + i\theta$ in accordance with the equation

$$\omega = \operatorname{Log} w/c, \tag{1.2}$$

where the principal value of the logarithm is used. We shall always assume, with Levi-Civita, that for all points of the flow

6. See [4] p. 101.

(1.3) $$\left| \frac{w - c}{c} \right| < 1.$$

Hence the function $\omega(\zeta)$ given by (1.2) is analytic for $|\zeta| < 1$.

In terms of the new variable $\omega = \tau + i\theta$, the condition (1.1) becomes

(1.4) $$\tfrac{1}{2} c^2 e^{2\tau} - gy = \text{const.}$$

We set $\zeta = \rho e^{i\varphi}$, so that $\varphi = 2\pi\, \Phi/c\lambda$ is a dimensionless velocity potential. Differentiating (1.4) with respect to $\varphi$ and simplifying, we get the boundary condition

(1.5) $$\frac{d\tau}{d\varphi} = -\gamma e^{-3\tau} \sin\theta \qquad \text{for } |\zeta| = 1$$

where $\gamma$ is the dimensionless parameter:

(1.6) $$\gamma = \lambda c \,/\, \pi c^2$$

The original problem of finding a steady flow $w(z)$ with period $\lambda$ satisfying the boundary condition (1.1) along $l$ and the condition $w = c$ at infinity is now replaced by the problem of finding, for a given positive number $\gamma$, a function $\omega(\zeta)$ satisfying the following conditions:

(1.7)

(a.) $\omega(\zeta) = \tau + i\theta$ is analytic for $|\zeta| < 1$;

(b.) $\omega(\zeta)$ is continuous for $|\zeta| \leq 1$;

(c.) $\omega(0) = 0$;

(d.) $\dfrac{d\tau}{d\varphi} = -\gamma e^{-3\tau} \sin\theta$ for $|\zeta| = 1$.

For convenience, we shall refer to this problem as Levi-Civita's problem.[7]

When the profile $l$ of the wave is symmetric about some vertical line the wave is called *symmetric*. If the $y$-axis is chosen to be along this vertical line, it can be shown that the condition

(1.8) $$\overline{\omega(\zeta)} = \omega(\bar{\zeta})$$

must hold for a symmetric wave. Conversely if (1.8) holds, it is obvious that the wave is symmetric with the $y$-axis of as an axis of symmetry.

7. For further details concerning the relationship of this problem to the original problem see [1], or [4], pp. 97-101.

*2. Another formulation of the problem.*

Following K. O. Friedrichs [2], p. 160–178, the above problem will be formulated in a somewhat different way. However, we do not assume, as Friedrichs does, that the solutions are symmetric in the sense that $\omega(\zeta) = \omega(\zeta)$

In the classical linearization of Airy the non-linear boundary condition $\frac{d\tau}{d\varphi} = -\gamma e^{-3\tau}\sin\theta$ is approximated by the nonear condition $\frac{d\tau}{d\varphi} = -\gamma\theta$ and this leads to the approximate solution $\omega(\zeta) = \mu\zeta^n$ where $\gamma$ is the positive integer $n$.

In trying to find true solutions "near" these approximate solutions, it is convenient to rewrite the principal boundary condition (1.5) in the form

$$\frac{d\tau}{d\varphi} + n\theta = n\theta - \gamma e^{-3\tau}\sin\theta \quad (|\zeta| = 1). \tag{2.1}$$

Our next objective is to formulate the problem in terms of an integral equation. To this end we study first the simpler problem

$$\frac{d\tau}{d\varphi} + n\theta = b(\varphi) \quad \text{along } |\zeta| = 1, \tag{2.2}$$

where for the present, $b(\varphi)$ is a given function. Later we shall replace $b(\varphi)$ by the right hand side of equation (2.1).

Let $H_0$ denote the space of all functions $\omega(\zeta)$ which are analytic for $|\zeta| < 1$, continuous for $|\zeta| \leq 1$, and vanish at $\zeta = 0$. With real numbers as scalars, $H_0$ is a vector space, and with the norm:

$$||\omega|| = max\ |\omega(\zeta)|, \quad : \quad |\zeta| \leq 1,$$

$H_0$ is a Banach space, as is easily verified.

Let $H_1$ denote the subclass of $H_0$ for which the two quantities

$$\lim_{p\to 1} \frac{\partial}{\partial\varphi} Re\,[\omega(\rho e^{i\varphi})] \quad \text{and} \quad \frac{d}{d\varphi} Re\,[\omega(e^{i\varphi})]$$

exist and are equal for $-\pi \leq \varphi \leq \pi$.

We then let $B$ denote the class of all real valued functions $b(\varphi)$ on $-\pi \leq \varphi \leq \pi$ which are continuous and satisfy the two conditions $b(\pi) = b(-\pi)$ and $\int_{-\pi}^{\pi} b(\varphi)d\varphi = 0$. With the usual maximum norm, $B$ is a Banach space

LEMMA 2.1 *For each $b \in B$, the function*

$$F(\zeta) = \frac{i\zeta}{\pi} \int_{-\pi}^{\pi} \frac{b(\varphi)}{e^{i\varphi} - \zeta}\, d\varphi \tag{2.3}$$

*is analytic for $|\zeta| < 1$ and $ImF(\rho e^{i\alpha})$ tends to $b(\alpha)$ as $\rho$ tends to one $(\rho < 1)$.*

*Proof:* If we put $\zeta=\rho e^{i\alpha}$ and perform some easy calculations we find that

$$\tau m F(\zeta) = -\tfrac{1}{2}\int_{-\pi}^{\pi} b(\varphi)\, d\varphi + \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-p^2)b(\varphi)}{1+p^2-2p\cos(\varphi-d)}\, d\varphi$$

The lemma now follows from the known properties of Poisson's integral, since the first integral is zero by hypothesis.

Let $\omega(\zeta)$ belong to $H_1$ and satisfy the boundary condition (2.2), where $\zeta = \rho e^{i\varphi}$ and $b(\varphi)$ is an element of $B$. Then observe that

$$\frac{d\tau}{d\varphi} + n\theta = - Im[\zeta^{n+1}\frac{d}{d\zeta}(\zeta^{-n}\omega)]_{\rho=1} \tag{2.4}$$

Now consider the function

$$G(\zeta) = F(\zeta) + \zeta^{n+1}\frac{d}{d\zeta}(\zeta^{-n}\omega) \tag{2.5}$$

where $F(\zeta)$ is defined by (2.3). By lemma 2.1, $G'(\zeta)$ is analytic for $|\zeta|<1$ and its imaginary part has a limit as $\rho \to 1$ - which vanishes identically on the circle $|\zeta| = 1$. Hence $G(\zeta)$ is constant for $|\zeta| < 1$, and since $G(0) = 0$ the value of the constant is zero. Therefore

$$F(\zeta) = -\zeta^{n+1}\frac{d}{d\zeta}(\zeta^{-n}\omega) = n\omega - \zeta\frac{d}{d\zeta}\omega \text{ for } |\zeta| < 1, \tag{2.6}$$

and $F^{(n)}(\zeta) = -\zeta\omega^{(n+1)}$. Hence $F^{(n)}(0) = 0$. Now by (2.3) the Taylor expansion of $F(\zeta)$ about $\zeta = 0$ is

$$F(\zeta) = \frac{i}{\pi}\sum_{n=1}^{\infty}\zeta^m\int_{-\pi}^{\pi} b(\varphi)e^{-im\varphi}d\varphi \tag{2.7}$$

Thus the condition $F^{(n)}(0) = 0$ is equivalent to

$$\int_{-\pi}^{\pi} b(\varphi)e^{-in\varphi}d\varphi = 0. \tag{2.8}$$

To obtain an integral representation for $\omega(\zeta)$, we divide equation (2.6) by $\zeta^{n+1}$ and integrate, obtaining

$$\omega(\zeta) = -\zeta^n\int_0^{\zeta}\frac{F(Z)}{Z^{n+1}}dZ + \mu\zeta^n \tag{2.9}$$

Then we substitute (2.7) into (2.9), making use of (2.8). The integration with respect to Z may be carried out term by term and the result is

$$(2.10) \qquad \omega(\zeta) = \frac{i}{\pi}\zeta^n\int_{-\pi}^{\pi} b(\varphi) \sum_{1\leq m\neq n} \frac{\zeta^{m-n}e^{-im\varphi}}{n-m}\, d\varphi + \mu\zeta^n$$

or

$$(2.11) \qquad \omega(\zeta) = \frac{i}{\pi}\int_{-\pi}^{\pi} b(\varphi) \sum_{m=1}^{n-1} \frac{\zeta^n e^{-im\varphi}}{m-m}\, d\varphi +$$

$$\frac{i}{\pi}\zeta^n\int_{-\pi}^{\pi} b(\varphi)e^{-in\varphi}\log(1-\zeta e^{-i\varphi})\,d\varphi + \mu\zeta_n,$$

where the principal value of the logarithm is used. From (2.10) it follows that $\mu = \omega^{(n)}(0)/n!$

These considerations establish the following result.

LEMMA 2.2: *Let b be an element of B. Then if $\omega(\zeta)$ is an element of $H_1$, which satisfies (2.2), it may be written in the form (2.10) or (2.11), where $\mu$ is a constant, and b must satisfy (2.8).*

In the following section we shall also need the converse of lemma 2.2 for the case $n = 1$. For $n = 1$ equation (2.11) becomes

$$(2.12) \qquad \omega(\zeta) = \frac{i}{\pi}\zeta\int_{-\pi}^{\pi} b(\varphi)e^{-i\varphi}\log(1-\zeta e^{-i\varphi})d\varphi + \mu\zeta.$$

LEMMA 2.3: *In the case $n = 1$, if $b \in B$ satisfies (2.8) and $\omega(\zeta)$ is defined by (2.12), then $\omega$ belongs to $H_1$ and satisfies the boundary condition (2.2)*

This lemma may be verified by straightforward but somewhat tedious calculations (see [2], pp. 188-178).

### 3. *Theorems of existence and uniqueness.*

In this section we study the non-linear integral equation obtained from (2.10) by replacing $b(\varphi)$ by the right member of (2.1), and obtain a locally unique solution depending on the two parameters $\gamma$ and $\mu$. Next, the cases where $n > 1$ are reduced to the basic case $n = 1$. For this case, $\mu$ may be taken as real without loss of generality. After establishing three properties of this two parameter family or solutions, it is shown that the auxiliary condition arising from (2.8) is satisfied if and only if a certain relationship holds between $\gamma$ and $\mu$. This leads us back to the solution of Levi-Cicita's problem.

LEMMA 3.1: *The transformation $b \to M_n[b]$ defined by*

$$(3.1) \qquad M_n[b] = \int_{-\pi}^{\pi} K_n(\zeta e^{-i\varphi})\, b(\varphi)\, d\varphi,$$

$$K(z) = \frac{i}{\pi} \sum_{1 \leq m \neq n} \frac{z^m}{n-m} = \frac{i}{\pi} \sum_{m=1}^{n-1} \frac{z^m}{n-m} + \frac{i}{\pi} z^n \log(1-z),$$

*is a linear bounded transformation of the Banach space B into the Banach space $H_0$, for each positive integer n.*

*Proof:* Clearly $M_n[b]$ lies in $H_0$ and is linear in $b$ for $b \in B$. To prove the boundedness, observe that

$$\int_0^{\pi/2} \log |\sin x|\, dx = -\frac{\pi}{2} \log 2$$

and that the principal value of $\log(1-e^{i(\alpha-\varphi)})$ is

$$\log 2 + \frac{i}{2}[\pi \operatorname{sgn}(\varphi-\alpha)] + \frac{i}{2}(\alpha-\varphi) + \log \left|\sin \frac{\varphi-\alpha}{2}\right|$$

Setting $\zeta = \rho e^{id}$ in (3.1) and using the above results we have

$$\| M[b] \| \leq \| b \| \left\{ \frac{1}{\pi} \sum_{m=1}^{n-1} \frac{1}{n-m} + (2+4\pi)\log 2 + 3\pi \right\}.$$

THEOREM 3.1: *Let*

$$b_n(\omega,\gamma) = n\theta(\varphi) - \gamma e^{-3\tau(\varphi)} \sin \theta(\varphi),$$

*where $\tau(\varphi) + i\theta(\varphi) = \omega(e^{i\varphi})$ and put $h_n(\zeta) = \zeta^n$, where n is a positive integer. To each n there corresponds a positive number $\delta_n$ such that, for every real $\gamma$ with $|\gamma - n| < \delta_n$ and every complex $\mu$, there is at most one solution of the equation*

$$\omega = M_n[b_n(\omega,\alpha)] + \mu b_n \tag{3.2}$$

*lying in the $\delta$-neighborhood of the origin of the space $H_0$. If in addition $|\mu| < \delta_n/2$, then there is exactly one solution $\omega = \omega_n(\zeta,\gamma,\mu)$ in this neighborhood.*

*Proof:* The partial derivatives of the function $b_n = n\theta - \gamma e^{-3\tau} \sin\theta$ with respect to $\tau$ and $\theta$ can be made arbitrarily small by taking $\tau$, $\theta$, and $|\gamma - n|$ sufficiently small. Also by lemma 3.1, the transformation $b \to M_n[b]$ is linear and bounded. Hence there exists a positive number $\delta_n$ such that

$$\| M_n[b_n(\omega,\gamma)] - M_n[b_n(\Omega,\gamma)] \| \leq \tfrac{1}{2} \| \omega - \Omega \|$$

whenever the numbers $|\gamma - n|$, $\|\omega\|$, $\|\Omega\|$ are each less than $\delta_n$. The statement of uniqueness follows easily from this Lipschitz condition.

To prove the existence, we first note that equation (3.2) is satisfied by taking $\mu = 0$, $\gamma = n$, and $\omega = 0$. Also, when $|\mu| < \delta_n/2$, $||\mu h_n|| < \delta_n/2$ and it follows by using the Lipschitz condition with $\Omega = 0$ that the transformation $\omega \to M_n[b_n(\omega,\gamma)] + \mu h_n$ takes the neighborhood $||\omega|| < \delta_n$ into itself. Hence we may use the method of successive approximations to establish the existence of the solution in this neighborhood.[8]

The next theorem shows that we may restrict ourselves to the case $n=1$.

THEOREM 3.2: *Using the notations of the preceding theorem, we have*

$$\omega_n(\zeta,\gamma,\mu) = \omega(\zeta^n,\gamma/n,\mu),$$

*for each real α in the neighborhood of $\gamma = n$ and each complex $\mu$ in the neighborhood of zero.*

*Proof: For a given value of n and k* $< 1$, let $\delta = \delta_n$ be chosen in accordance with theorem 3.1, and let $\epsilon = min(\delta_1, \delta_n)$. Now choose $\gamma$ so that $|\delta - n| < \epsilon$ and choose $\mu$ so that $|\mu| < \epsilon/2$. Putting $\gamma_1 = \gamma/n$, we have $|\gamma_1 - 1| < \epsilon/n < \delta_1$, and $|\mu| < \delta(1-k)$. Hence by theorem 3.1 there is a unique solution $\omega = \omega_1(\zeta,\gamma,\mu)$ of equation (3.2) for the case $n = 1$. Writing out this equation (cf. (3.1)) and replacing $\zeta$ by $\zeta^n$ we get

$$\omega_1 = -\frac{i}{\pi}\int_{-\pi}^{\pi} b_1[\omega_1(\varphi),\gamma_1]\sum_{j=1}^{\infty}\frac{\zeta^{n(j+1)}\, e^{-i(j+1)\varphi}}{j}\,d\varphi + \mu\zeta,$$

where $\omega_1(\varphi) = \tau_1(\varphi) + i\,\theta_1(\varphi)$ has period $2\pi$. The integrand being periodic, we may replace the limits by $-n\pi$ and $n\pi$ if we simultaneously divide by $n$. Then we are ready to make the change of variable $\varphi = n\varphi_1$ and obtain:

$$\omega_1(\zeta^n) = -\frac{i}{\pi}\int_{-\pi}^{\pi} nb_1[\omega(n\varphi_1)n,\gamma_1]\sum_{j=1}^{\infty}\frac{(\zeta e^{-i\varphi_1})^{nj+n}}{nj}\,d\varphi_1 + \mu\zeta^n.$$

A change of the index of summation by putting $m = nj + n$ gives

$$\omega_1(\zeta^n) = -\frac{i}{\pi}\int_{-\pi}^{\pi} nb_1[\omega_1(n\varphi),\gamma_1]\sum\frac{(\zeta e^{-i\varphi})^m}{m-n}\,d\varphi + \mu\zeta^n,$$

in which the index $m$ of summation varies over the proper multiples $2n$, $3n$. ... of $n$. However, since $b_1(\varphi) = b_1[\omega_1(\varphi),\gamma_1]$ is continuous and has the period $2\pi$, it follows that $b_1(n\varphi)$ is orthogonal to the functions $e^{mi\varphi}$, in case $m$ is *not* a multiple of $n$. Thus these other values of $m$ may also be included in the summation, and it follows from (3.1) that $\omega = \omega_1(\zeta^n, \gamma/n, \mu)$ is a solution of (3.2) in which

8. See [3], pp. 133-134.

$$b_n(\omega,\gamma) = n\theta_1(n\varphi) - ,\gamma e^{-3\tau_1(n\varphi)} \sin \theta_1(n\varphi) = nb_1[\omega_1(n\varphi), \delta_1]$$

But this solution is unique and the theorem follows.

From now on we limit ourselves to the basic case $n = 1$, so that $\gamma$ will be selected in the neighborhood of $\gamma = 1$. For this case $\mu = \omega'(0)$ and by a proper choice of the origin in the $z$-plane, *we may take $\mu$ real without any loss of generality.*[9]

THEOREM 3.3: *The solution $\omega = \omega(\zeta,\gamma,\mu)$ of the equation*

$$\omega = M_1[b_1(\omega,\gamma)] + \mu h_1, \tag{3.3}$$

*whose existence was established in theorem* (3.1) *has the following properties. For every real value of $\gamma$ and of $\mu$ in suitable neighborhoods of $\gamma = 1$ and $\mu = 0$,*

(i) *$\omega(\zeta,\gamma,\mu)$ has derivatives of all orders with respect to $\delta$ and $\mu$;*

(ii) *$\omega(\zeta,\gamma,\mu) = \mu\hat{\omega}(\zeta,\gamma,\mu)$ where $\hat{\omega}$ begongs to $H_0$ and has derivatives of all orders with respect to $\alpha$ and $\mu$;*

(iii) *$\overline{\omega(\zeta,\gamma,\mu)} = \omega(\overline{\zeta},\gamma,\mu)$, so that all the solutions are symmetric.*

*Proof:* Since $M_1[b]$ is bounded and linear, and since $b_1(\omega,\gamma) = \theta - \gamma e^{-3\tau}$ $\sin\theta$ is indefinitely differentiable in $\theta$, $\tau$, $\alpha$, it is easily seen that the right member of (3.3), looked upon as an element of $H_0$, has Fréchet differentials of all orders with respect to the triple $(\omega,\gamma,\mu)$. Moreover, each of these differentials is continuous in all of its variables uniformly with respect to the increments $\delta\omega$, $\delta\gamma$, $\delta\mu$ for $\|\delta\omega\| \le 1$, $|\delta\gamma| \le 1$, $|\delta\mu| \le 1$. Consequently, it can be shown that the right member of (3.3) is of class $C^{(n)}$, in the sense of Hildebrandt and Graves [3], for each positive integer $n$, and the property (i) now follows by applying their theorem IV, p. 150.[10]

To establish properties (ii) and (iii) we define a sequence of approximations $\omega_m$ by

$$\omega_0 = 0;\ \omega_{m+1} = M_1[\theta_m - \gamma e^{-3\tau_m} \sin\theta_m] + \mu h_1, \tag{3.4}$$

where $\omega_m = \tau_m + i\theta_m$. Trivially, $\omega_0 = 0$ for $\mu = 0$. Making the induction assumption that $\omega = 0$ for $\mu = 0$, it follows from (3.4) that $\omega_{m+1} = 0$ for $\mu = 0$. Hence the limit

$$\omega(\zeta,\gamma,\mu) = \lim_{m\to\infty} \omega_m(\zeta,\gamma,\mu),$$

taken in the sense of the norm in $H_0$, also vanishes for $\mu = 0$. From this result together with property (i), property (ii) is easily established.

9. See [4], p. 101.

10. One can also establish property (i) by using the method of successive approximations directly.

Property (iii) can be proved inductively in a similar manner, if we observe thaf $\omega_m(\bar{\zeta}) = \overline{\omega_m(\zeta)}$ implies that $\theta_m$ is odd and $\tau_m$ is even in $\varphi$, so that

$$b_m(\varphi) = \theta_m - \gamma e^{-3\tau_m} \sin \theta_m$$

is odd.

Our next objective is to show that the auxiliary condition (2.8) for $n = 1$ can be satisfied if and only if a certain relationship holds between the parameters $\mu$ and $\gamma$. Substituting the solution $\omega(\zeta, \gamma, \mu)$ of (3.3) into (2.8), where $b = \theta - \gamma e^{-3\tau} \sin \theta$, gives the relation [11]

$$(3.5) \qquad V(\gamma, \mu) = \int_{-\pi}^{\pi} \sin \varphi \ \{\theta(\varphi) - \gamma e^{-3\tau(\varphi)} \sin \theta \ (\varphi)\} \, d\varphi = 0.$$

In order to show that this equation can be solved for $\gamma$ near $\mu = 0$, we first divide by $\mu$, obtaining the equation

$$(3.6) \qquad \hat{V}(\gamma, \mu) = \int_{-\pi}^{\pi} \sin \varphi \ \{\hat{\theta} - \gamma e^{-3\mu\hat{\tau}} \frac{\sin \mu\hat{\theta}}{\mu}\} \, d\varphi = 0.$$

Now the function $\hat{\omega}(\zeta, \gamma, \mu)$ satisfies the equation

$$(3.7) \qquad \hat{\omega} = M_1[\hat{\theta} - \gamma \, e^{-3\mu\hat{\tau}} \frac{\sin \mu\hat{\theta}}{\mu}] + h_1,$$

from which it follows that $\hat{\omega}(\zeta, 1, 0) = h_1(\zeta) = \zeta$. Differentiating $V(\gamma, \mu)$ under the integral sign with respect to $\gamma$ and setting $\gamma = 1$ and $\mu = 0$, we get

$$\hat{V}_\gamma(1,0) = \int_{-\pi}^{\pi} \sin \varphi \{\hat{\theta}_\gamma - \theta - \hat{\theta}_\gamma\}_{\substack{\gamma=1\\ \mu=0}} d\varphi = - \int_{-\pi}^{\pi} \sin^2\varphi \, d\varphi = - \pi$$

Clearly equation (3.6) is satisfied for $\alpha = 1$ and $\mu = 0$. Hence by the ordinary implicit function theorem, (3.6) may be solved uniquely for $\gamma$ in terms of $\mu$, in the form $\gamma = \gamma(\mu)$, this function being defined near $\mu = 0$. We summarize our results in the following theorem.

THEOREM 3.4: *There exist positive numbers $\delta$, $\eta$ such that for each value of the real parameter $\mu = \bar{\omega}'(0)$ satisfying $|\mu| < \eta$, there exists a unique solution $\gamma$, $\omega$ of Levi-Civita's problem satisfying $|1 - \gamma| < \delta$, $||\omega|| < \delta$. This solution is symmetric in the sense that $\omega(\bar{\zeta}) = \overline{\omega(\zeta)}$.*

*Proof:* To establish the uniqueness, let $\gamma$, $\omega$ be a solution of Levi-Civita's problem. Then since $\omega = 0$ for $\zeta = 0$, it follows by the mean value theorem for harmonic functions that $\int_{-\pi}^{\pi} \theta(\varphi) \, d\varphi = 0$. Consequently

11. Since $\omega$ has the symmetry property (iii), the function $b = \theta - \gamma e^{-3\tau} \sin\theta$ is odd in $\varphi$, so that relation (2.8) reduces to (3.5).

$b \in B$, where $b = \theta - \gamma e^{-3\tau} \sin \theta$. By lemma 2.2, $\omega$ satisfies (2.12) or equivalently (3.3) and $b$ satisfies (2.8). By theorem 3.1 there is a unique solution of (3.3) providing $\gamma$ and $\mu$ are in the neighborhood of $\gamma = 1$ and $\mu = 0$ and $\omega$ is in the neighborhood of the origin in the space $H_0$. Since $b$ satisfies (2.8), $\gamma$ and $\mu$ satisfy (3.6), so that $\gamma = \gamma(\mu)$ is uniquely determined by $\mu$.

The existence is proved by reversing this argument, beginning with the solution $\omega = \omega_1(\zeta, \gamma, \mu)$ given by theorem 3.1, and using lemma 2.3 in place of lemma 2.2.

Let the choice of $z = 0$ on the profile $h$ be made so that $\mu$ is real and *negative*, as described in section one. Since $\omega$ is symmetric by theorem 3.3, it follows that the origin $z = 0$ is at a crest or else at a trough of the wave. We next show that the origin $z = 0$ is located at a crest when $\mu < 0$.

Putting $\tau_0 = Re\ \omega(1)$, we see that since the solution $\omega(\zeta)$ is determined by the parameter $\mu$, $\tau_0$ is a function of $\mu$ alone in the neighbor- of $\mu = 0$. Now using theorem 3.3 with $\gamma = \gamma(\mu)$, we see that $\tau_0$ is a continuously differentiable function of $\mu$ alone, and that $\tau_0(\mu) = \mu\hat{\tau}_0(\mu)$, where $\hat{\tau}(\mu) = Re\ \hat{\omega}(1)$ is also continuously differentiable in $\mu$. It follows that $\tau_0'(0) = \hat{\tau}_0(0) = 1$, since $\omega = \zeta$ for $\mu = 0$. Since $\tau(0) = 0$, we see that $\tau_0$ is negative since $\mu$ is negative. Thus $w = ce^{\tau}$ is real and less than $c$ at $z = 0$, which therefore cannot be a trough and must be a crest.[12]

The argument just concluded in the last paragraph also shows that *we may use* $\tau_0$ *or the velocity* $w_0 = ce^{\tau_0}$ *at a crest as the basic parameter in place of* $\mu$, *if desired,* in stating a uniqueness theorem analogous to theorem 3.4.

THEOREM 3.5: *There exist positive numbers* $\delta$, $\epsilon$ *such that for each* $\alpha$ *satisfying* $0 < 1 - \gamma < \delta$, *there exists a unique solution* $\omega = \omega(\zeta)$ *of Levi-Civita's problem satisfying the inequalities* $||\omega|| < \epsilon$ *and* $\omega'(0) < 0$.

*Proof:* By theorem 3.1 for the case $n = 1$, there exists a $\delta > 0$ such that equation (3.3) has a unique solution $\omega = \omega_1(\zeta, \alpha, \mu)$ satisfying

$$|\gamma - 1| < \delta_1, \quad |\mu| < \tfrac{1}{2}\delta_1, \quad ||\omega|| < \delta_1,$$

and by theorem 3.3, $\omega_1(\zeta, \gamma, \mu)$ has derivatives of all orders with respect to $\alpha$ and $\mu$. By expanding $\omega_1(\zeta, \gamma, \mu)$ in powers of $\gamma$ and $\mu$ by Taylor's formula with remainder, determining the first few coefficients in this expansion by substitution into (3.3) and then substituting the resulting expansion into the auxiliary equation (2.8), Friedrichs ([2], pp. 179–182) has shown that (2.8) or equivalently (3.6) reduces to [13]

12. See [1], section 9.

13. In equation (4), p. 182 of [2], the coefficient of $c^2$ should be -1. Equation (3.8) also agrees with Levi-Civita's relations (18), p. 312 of [1].

$$(3.8) \qquad 1 - \gamma = \mu^2 + 0\,[\mu^3].$$

It follows that (3.8) can be solved uniquely for $\mu^2$ in terms of $\gamma$, and since by hypotheses $\mu = \omega'(0) < 0$, for $\mu$ in terms of $\gamma$. More precisely, there exist positive numbers $\delta' < \delta_1$ and $\epsilon < \frac{1}{2}\delta_1$ such that (3.8) has a unique solution $\mu = \mu(\gamma)$ under the restrictions $0 < 1 - \gamma < \delta'$ and $0 < -\mu < \epsilon$. By lemma 2.3 the function $\omega = \omega_1(\zeta, \gamma, \mu(\gamma))$ is a solution of Levi-Civita's problem. By restricting $\gamma$ even more if necessary, say $0 < 1 - \gamma < \delta < \delta'$, we can insure that $||\omega|| < \epsilon$. Since $|\omega'(0)| \leq ||\omega||$, it follows that $-\mu(\gamma) = -\omega'(0) < \epsilon$. Hence our solution is unique under the conditions of the theorem.

In terms of our original quantities, putting $U = u - c$, $V = v$, we have:

THEOREM 3.6: *There exist positive numbers $\delta$ and $\epsilon$ such that, for every value of the wavelength $\lambda$ and the propagation velocity c satisfying* $1 < 1 - (g\lambda/2\pi c^2) < \delta$, *there exists a unique non-trivial wave of* permanent *type in an infinitely deep channel whose (actual) particle velocity vector (U,V) satisfies $U^2 + V^2 < \epsilon^2$. There are no such waves satisfying this restriction on the velocity and*

$$0 \leq (g\lambda/2\pi c^2) - 1 < \delta,$$

*except the trivial one for which $U = V = 0$.*

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# A NOTE CONCERNING HOMOGENEOUS POLYNOMIALS

J. C. Morelock and N. C. Perry

## 1. *Introduction*

In classical algebraic geometry there has been considerable interest in specific instances of homogeneous polynomial surfaces $F(x_1, x_2, x_3, x_4) = 0$ of prime degree $p$, which are invariant under the cyclic transformation $T$ defined by:

$$(x_1', x_2', x_3', x_4') = (x_1, Ex_2, E^2x_3, E^3x_4), \quad \text{where } E^p = 1.^*$$

For example, it is a known result, obtained apparently by actual trial, that the homogeneous polynomial of degree 5 with the largest number of terms which is invariant under $T$ is

$$F = x_1^5 + x_2^5 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_4^2 + x_1^3x_3x_4 + x_3^5 + x_2x_3^3x_4$$
$$+ x_2^2x_3x_4^2 + x_1x_3^2x_4^2 + x_1x_2x_4^3 + x_4^5.$$

This paper proposes to give a general treatment of these instances by characterizing all such polynomials for any prime $p > 3$, by a method which employs elementary number theory only.

We note, to begin with, that a surface $F(x_1, x_2, x_3, x_4) = 0$, is invariant if each of its terms $x_1^a\, x_2^b\, x_3^c\, x_4^d$ is transformed by $T$ into the expression $E^s\, x_1^a\, x_2^b\, x_3^c\, x_4^d$, where $s$ is of the same residue class (modulus $p$) for all terms. This situation will be referred to hereafter as $s$-invariance.

## 2. *Necessary and sufficient conditions for s-invariance:*

Required conditions on $a$, $b$, $c$, and $d$ are:

$$a + b + c + d = p$$

$$b + 2c + 3d = s' + kp \qquad (0 \le s' \le p - 1)$$

Since $a$, $b$, $c$, and $d$ are non-negative integers where each is at most equal to $p$, it is clear on inspecting the last equation that $k$ can take on only the values 0, 1, 2, and 3.

Diophantine solutions of the two equations are easily obtained by elementary methods as:

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* L. Godeaux's *Geometrie Algebrigue*

$$a = n \qquad d = r$$

$$b = r - 2n - s' + (2 - k)p$$

$$c = n - 2r + s' + (k - 1)p$$

If $k = 0$ we have the four inequalities:

$$(1) \quad 0 \leq n \leq p$$

$$(2) \quad 0 \leq r - 2n - s' + 2p$$

$$(3) \quad 0 \leq n - 2r + s' - p$$

$$(4) \quad 0 \leq r \leq p$$

From inequality (3) we have $2r - s' + p \leq n$ and from (2) it follows that $n \leq \dfrac{r - s' + 2p}{2}$. Thus $n$ must lie within the range defined by $2r - s' + p \leq n \leq \dfrac{2p + r - s'}{2}$. Since from (3) $4r - 2s' + 2p \leq 2n$ we may replace $2n$ in (2) by $4r - 2s' + 2p$ obtaining $0 \leq r - 4r + 2s' - 2p - s' + 2p$ which simplifies into $3r \leq s'$.

Hence, to obtain an invariant term of degree $p$, necessary restrictions on $r$ and $n$ are (for the case $k = 0$) first that $r$ be chosen on the range $0 \leq r \leq s'/3$, and then $n$ be chosen on the range $2r - s' + p \leq n \leq \frac{1}{2}(2p + r - s')$.

The argument by which these restrictions on $n$ and $r$ were obtained from inequalities (2) and (3) is clearly reversible. Hence, since the obtained solutions to the Diophantine equations are the only possible solutions, the conditional inequalities $0 \leq r \leq s'/3$, $2r - s' + p \leq n \leq \frac{1}{2}(2p + r - s')$, constitute with the equations determining $b$ and $c$, a set of necessary and sufficient conditions for an invariant term of degree $p$, for the case $k = 0$.

By similar arguments, necessary and sufficient conditions can be obtained for the cases, $k = 1$, $k = 2$, and $k = 3$.

We summarize all these results in the following:

Theorem I: *Necessary and sufficient conditions that a polynomial term $x_1^a x_2^b x_3^c x_4^d$ of degree $p = 3$ shall be s-invariant under $T$ are –*

1. $a = n$, $b = r - 2n - s' + (2 - k)p$,
   $c = n - 2r + s' + (k - 1)p$, $d = r$
   (where $k = 0, 1, 2,$ or $3$)

2. If $k=0$, $0 \le r \le s'/3$, $2r - s' + p \le n \le \frac{1}{2}(2p + r - s')$

If $k=1$, $0 \le r \le s'/2$, $0 \le n \le \frac{1}{2}(p + r - s')$

$\frac{1}{2}(s' + 1) \le r \le 1/3\ (p + s')$, $2r - s' \le n \le \frac{1}{2}(p + r - s')$

If $k=2$, $s' \le r \le \frac{1}{2}(p + s')$, $0 \le n \le \frac{1}{2}(r - s')$

$\frac{1}{2}(p + s' + 1) \le r \le 1/3\ (2p + s')$, $2r - s \le n \le \frac{1}{2}(r - s\ )$

If $k=3$, $r = p$, $n = 0$ $\quad (b = c = s' = 0)$

*3. The number of terms in the most general invariant polynomial for any* $s'$.

By an intricate analysis the possible choices of the pair $(n, r)$ satisfying Theorem I, can be enumerated for any $s'$. The method employed [5], uses ordinary number theory only, but involves an elaborate bread-down into cases where all possible combinations are considered of $(r - s)$ even or odd, $p$ of the form $6\alpha \pm 1$, $s'$ of the form $6\beta$, $6\beta \pm 1$, $6\beta \pm 2$, $6\beta + 3$, for each of the first three sets of conditional inequalities in Theorem I.

We note that for a given $s'$ distince pairs $(n, r)$ produce different polynomial terms, because $n$ is the exponent of $x_1$ and $r$ is the exponent of $x_4$. Then since any two of the inequality sets are non-overlappong in the least one of the variables $n$ and $r$, any duplication of polynomial terms in the enumeration is impossible. The same pair $(n, r)$ may arise for different $s'$ but in this instance the exponents of $x_2$ and $x_3$ will be different, since $b$ and $c$ are strictly monotonic functions of $s'$.

The results obtained can be summarized –

Theorem II: *For each* $s'$ *such that* $1 \le s' \le p - 1$ *there are* $\frac{p^2 + 6p + 11}{6}$ *invariant polynomial terms. For* $s' = 0$, *there is one additional term as noted under the case* $k = 3$, *or* $\frac{p^2 + 6p + 17}{6}$ *terms.*

These results check with the classical theorem stating the number of terms in the entire homogeneous polynomial* of degree $p$, for:

$$\frac{p^2 + 6p + 17}{6} + (p - 1)\,\frac{(p^2 + 6p + 11)}{6} = \frac{(p + 1)(p + 2)(p + 3)}{6}$$

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* **Snyder and Sisam**, *Analytic Geometry of Space*.

The method described in this paper permits a classification of the general homogeneous polynomial into $p$ sets, where to each set there corresponds an invariant surface. Any subset of any one of these sets also yields an invariant surface.

Thus, the approach used makes it possible to form invariant polynomials without trial and error, and also affords a new way of counting the terms in the most general homogeneous polynomial of prime degree $p$.

### 4. *Conclusion; A numerical example.*

We illustrate the application of Theorem I and II for the case $p = 5$. Geometric usage of such developments can be found in Dessart [1], and Hutcherson [3] and [4], for polynomials of degree 5, 7, and 11.

For $p = 5$ the inequalities of Theorem I yield the following tabulation of pairs $(n,r)$:

| $s' = 0$ | | | $s' = 1$ | | | $s' = 2$ | | | $s' = 3$ | | | $s' = 4$ | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $r$ | $n$ | $k$ | $r$ | $n$ | $k$ | $r$ | $n$ | $k$ | $r$ | $n$ | $k$ | $r$ | $n$ |
| 0 | 0 | 5 | 0 | 0 | 4 | 0 | 0 | 3,4 | 0 | 0 | 2,3 | 0 | 0 | 1,2,3 |
| | 0 | 0,1,2 | | 0 | 0,1,2 | | 0 | 0,1 | | 1 | 4 | 1 | | 3 |
| 1 | 1 | 2,3 | 1 | 1 | 1,2 | 1 | 1 | 0,1,2 | | 0 | 0,1 | | 0 | 0 |
| | 0 | 0 | | 2 | 3 | | 2 | 2 | 1 | 1 | 0,1 | | 1 | 0,1 |
| | 1 | 0 | | 1 | 0 | | 2 | 0 | | 2 | 1,2 | 1 | 2 | 0,1 |
| 2 | 2 | 0,1 | 2 | 2 | 0 | 2 | 3 | 0 | 2 | 3 | 0 | | 3 | 2 |
| | 3 | 1 | | 3 | 0,1 | | 4 | 1 | | 4 | 0 | 2 | 4 | 0 |
| 3 | 5 | 0 | | | | | | | | | | | | |

As described in Theorem II this tabulation yields a 12-term polynomial for the case $s' = 0$, and 11-term polynomials for each of the cases $s' = 1,2,3,$ and 4. Thus, in all, we have 12 + 44 or 56 terms in the most general homogeneous polynomial, as described by the general expression $(p + 1)(p + 2)(p + 3)/6$ when evaluated for $p = 5$.

These five polynomials are listed below as computed through use of the $(n,r)$ tabulation and the expressions for $b$ and $c$ in Theorem II:

$$F_0;\quad x_1^5 + x_2^5 + x_1x_2^3x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_4 + x_1^3x_3x_4 + x_3^5 + x_2x_3^3x_4 + x_2^2x_3x_4^2 + x_1x_3^2x_4^2 + x_1x_2x_4^3 + x_4^5$$

$$F_1;\quad x_1^4x_2 + x_2^4x_3 + x_1x_2^2x_3^2 + x_1^2x_3^3 + x_1x_2^3x_4 + x_1^2x_2x_3x_4 + x_1^3x_4^2 + x_3^4x_4 + x_2x_3^2x_4^2 + x_2^2x_4^3 + x_1x_3x_4^3$$

$$F_2;\quad x_1^3x_2^2 + x_1^4x_3 + x_2^3x_3^2 + x_1x_2x_3^3 + x_2^4x_4 + x_1x_2^2x_3x_4 + x_1^2x_2^2x_4 + x_1^2x_2x_4^2 + x_3^3x_4^2 + x_2x_3x_4^3 + x_1x_4^4$$

$$F_3; \quad x_1^2 x_2^3 + x_1^3 x_2 x_3 + x_1^4 x_4 + x_2^2 x_3^3 + x_1 x_3^4 + x_2^3 x_3 x_4 + x_1 x_2 x_3^2 x_4 + x_1 x_2^2 x_4^2 + x_1^2 x_3 x_4^2 + x_3^2 x_4^3 + x_2 x_4^4$$

$$F_4; \quad x_1 x_2^4 + x_1^2 x_2^2 x_3 + x_1^3 x_3^2 + x_1^3 x_2 x_4 + x_2 x_3^4 + x_2^2 x_3^2 x_4 + x_1 x_3^3 x_4 + x_2^3 x_4^2 + x_1 x_2 x_3 x_4^2 + x_1^2 x_4^3 + x_3 x_4^4$$

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# Certain Non-factorable Polynomials

Doyne Holder

Teachers of algebra in secondary schools find it convenient to be able to determine quickly if a trinomial of the general type $ax^2 + bx + c$ is or is not factorable. A proof is given here that a certain type is non-factorable.

Theorem: *In the general trinomial $ax^2 + bx + c$, if $a$, $b$, and $c$ are all odd integers, then there are no linear integral factors of $ax^2 + bx + c$.*

Proof: If $ax^2 + bx + c$ is factorable, let these factors be such that

$$ax^2 + bx + c = (mx + r)(nx + s).$$

Then

$$(1) \quad mn = a$$

$$(2) \quad rs = c$$

$$(3) \quad ms + nr = b$$

Since both $a$ and $c$ are odd, it is trivial that $m$, $n$, $r$, and $s$ are all odd, hence all non-zero. Furthermore, the products, $ms$ and $nr$, are both odd - each being the product of two odd numbers.

By (3) $ms + nr = b$, and $b$ is odd. But the sum of two odd numbers cannot be odd; hence in the case where $a$, $b$, and $c$ are all odd there are no linear integral factors of $ax^2 + bx + c$.

The above theorem can be extended to polynomials of higher degree by altering the proof only slightly as shown here.

Theorem: *If $a(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is a polynomial whose coefficients are integers and if $a_n$, $a_0$, and $a(1)$ are all odd, then there are no linear integral factors of $a(x)$.*

Proof: If there is a linear integral factor of $a(x)$ let it be

$$d(x) = mx + k$$

It is a trivial that $m$ must divide $a_n$ and $k$ must divide $a_0$. Since both $a_n$ and $a_0$ are odd, then both $m$ and $k$ are odd, hence both non-zero.

Furthermore $d(1)$ divides $a(1)$ and as $a(1)$ is odd, $d(1)$ must be odd.

But $d(1) = m + k$, both $m$ and $k$ being odd, and the sum of two odd numbers cannot be odd.

Therefore, if $a_n$, $a_0$, and $a(1)$ are all odd, there can be no linear integral factors of $a(x)$.

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# THE LUCKY NUMBER THEOREM

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The lucky numbers of Ulam resemble prime numbers in their apparent distribution among the natural numbers and with respect to the kind of sieve that generates them [1]. It is therefore of interest to investigate their properties, bearing in mind the analogies to prime number theory.

The lucky numbers are defined by the following sieve. If $S_n$ is an infinite sequence of natural numbers $t_{n,m}$ $(m = 1, 2, 3, \ldots)$, one obtains $S_{n+1}$, for $n > 1$, from $S_n$ by removing every $t_{n,m}$ for which $t_{n,n}$ divides $m$. $S_2$ is the sequence $2, 3, 5, 7, 9, \ldots$ of the number 2 followed by the odd integers in increasing order of magnitude. $S_1$ is the sequence of natural numbers. The sequence of lucky numbers is $S = \lim_{n\to\infty} S_n$, that is, $2, 3, 7, 9, 13, 15, 21, \ldots$ . This definition differs trivially from Ulam's.

Two properties are basic for the investigation of saymptotic properties of lucky numbers. By the definition if $s_m$ represents the $m$-th lucky number, then

$$(1) \qquad s_m = t_{n,m} \quad \text{for all } m < s_n.$$

This follows from the fact that $t_{n,s(n)}$ is the first number that will be removed from $S_n$ in forming $S_{n+1}$, and is at the same time less than any number to be removed later on. Also by the definition, if $R(n,x)$ is the number of numbers not greater than $x$ in $S_n$, and $[x]$ denotes the greatest integer in $x$, then

$$(2) \qquad R(n,x) = R(n-1,\ x) - \frac{R(n-1,\ x)}{s_{n-1}}, \quad n \geq 2.$$

This fundamental recurrence relation has the following solution, in which $\{x\}$ denotes the fractional part of $x$ and

$$\sigma_n = (1 - 1/2)(1 - 1/3)(1 - 1/7) \ldots (1 - 1/s_{n-1}),$$

$$(3) \qquad R(n,x) = [x]\,\sigma_n + \sum_{i=2}^{n} \frac{\sigma_n}{\sigma_i} \frac{R(i-1,\ x)}{ss_{i-1}}, \quad n \geq 2,$$

$$= [x]\,\sigma_n + E(n,x).$$

$$(4) \qquad 0 \leq (n,x) < n.$$

The properties is $S$ now develop by a series of stages. The first is to find bounds for $s_n$. If one puts $R(n, s_{n+r}) = n + r$, which by (1) one may do for $0 \le r < s_n - n$, then

(5) $$s_n = \frac{n}{\sigma_n}$$

by letting $r = 0$. From (5)

$$\frac{\sigma_{n+1}}{\sigma_n} = 1 - \frac{1}{s_n} \le 1 - \frac{\sigma_n}{n}, \quad n \ge 2.$$

Putting $\rho_n = 1/\sigma_n$

$$\rho_{n+1} - \rho_n \ge \frac{\rho_{n+1}}{n\rho_n} > \frac{1}{n}, \quad n \ge 2.$$

Summing from 2 to $n - 1$, one obtains $\rho_n - \rho_2 > \sum_{t=2}^{n-1} \frac{1}{t}$ which implies $\rho_n > \log n$, or

(6) $$\sigma_n < \frac{1}{\log n}, \qquad n \ge 2$$

In $R(n, s_{n+r}) = n + r$, $(0 \le r < s_n - n)$, one may set $r = n$ and $r = n - 1$, since $s_n \ge 2n$ for $n > 2$, which gives, by (3)

$$2n = s_{2n}\, \sigma_n + E(n, s_{2n})$$

$$2n - 1 = s_{2n-1}\, \sigma_n + E(n, s_{2n-1}).$$

Hence, by (4) and (6), for $n > 2$,

(7) $$s_{2n} > n \log n \qquad s_{2n-1} > (n - 1) \log n.$$

It is now possible to show that the remainder term $E(n,x)$ is $o(n)$ when $x = s_n$. For from (1) and (3),

(8) $$R(n, s_n) = s_n\, \sigma_n + E(n, s_n) = n.$$

Let $\alpha(n)$ be the integer defined by $s_{\alpha(n)} \ge n$ and $s_{\alpha(n)-1} < n$. By (7)

(9) $$\alpha(n) < \frac{3n}{\log n}, \qquad n > n_0.$$

Now split the sum $E(n, s_n)$ into two parts $E_1$ and $E_2$, where

(10) $$E_1 = \sum_{i=2}^{3n/\log n} \frac{\sigma_n}{\sigma_i} \, \frac{R(i - 1, s_n)}{s_{i-1}}$$

(11) $$E_1 = \sum_{i=3n/\log n}^{n} \frac{\sigma_n}{\sigma_i} \frac{R(i-1,\ s_n)}{s_{i-1}}$$

Clearly

(12) $$E_1 = 0 \frac{n}{\log n}$$

In $E_2$, because of (1), put all $R(i-1,\ s_n) = R(\alpha(n),\ s_n) = n$, so that by (7)

= (13) $$= E_2 = 0 \sum_{i=3n/\log n}^{n} \frac{n}{i \log i} = 0 \frac{n \log \log n}{\log n} .$$

It is now possible to write for $n \geq 2$,

(14) $$\frac{\sigma_{n+1}}{\sigma_n} = 1 - \frac{1}{s_n} = 1 - \frac{\sigma_n}{n + o(n)}$$

Again using the substitution $\rho_n = 1/\sigma_n$ , one obtains

$$\rho_{n+} - \rho_n \equiv \frac{\rho_{n\pm 1}}{n\rho_n} + 0 \frac{1}{n} \frac{\rho_{n\pm 1}}{\rho_n} .$$

By summation $\rho_n = \log n + o(\log n)$ and, therefore

(15) $$\sigma_n \sim \frac{1}{\log n}$$

and from this, (1), and (3)

(16) $$s_n = \frac{n + o(n)}{\sigma_n} \sim n \log n .$$

(15) is the analogue of the Merten's theorem for prime numbers, and (16) is the analogue of the prime number theorem.

These results, especially (16), confirm a conjecture of one of the authors based on stochastic arguments (2). They support the observation that the asymptotic distribution of prime numbers is not, except in details, a consequence of their primality, but characteristic of a wide class of sieve-generated sequences, of which the lucky numbers are an example.

By using the results recursively in (14), S. Chowla has shown that the asymptotic value of (15) can be improved to

(17) $$s_n = n \log n + \frac{n}{2} (\log \log n)^2 + o[n(\log \log n)^2].$$

Since the corresponding result for prime numbers (where $p_n$ is the $n$-th prime number) is

$$p_n = n \log n + n \log \log n + o(n \log \log n), \tag{18}$$

it follows, with only a finite number of exceptions, that $s_n > p_n$. With necessary calculations, this presumably will confirm Ulam's conjecture $s_n > p_n$ for all $n$.

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# CRITICAL CURVES IN SEISMIC EXPLORATION

H. G. Helfenstein

## 1. Foreword

In seismic prospecting a charge is exploded at a point $O$ on the surface of the earth and the time for the seismic waves to return to the surface is measured in various points in the neighborhood of $O$. The velocity of $s$ of the waves in the uppermost layer down to the reflecting discontinuity surface $D$ can be measured separately. By means of these data we can plot points of the "time-distance surface" defined as follows.

Introduce a system of coordinates with $O$ as origin, $z$-axis pointing vertically downwards, any $x,y$ plane coinciding with the surface of the earth. At every point $F = (u,v,o)$ we measure a value of the travel time $T(u,v)$ and a total distance $w = sT$ travelled by the wave. The surface with equation $w = sT = \Omega(u,v)$ is the "time - distance surface" $S$.

It is easy to see that the fictitious surface $S$ obtained from our measurements and the physical surface $D$ are connected by a contact transformation $T$ [1]. According to the law of reflection the ellipsoid $E$ of revolution with foci $O = (o,o,o)$ and $F = (u,v,o)$ and major axis $w = \Omega(u,v)$ is tangent to $D$. The equation of $E$ is given by

$$(1) \qquad w = \sqrt{x^2 + y^2 + z^2} + \sqrt{(u-x)^2 + (v-y)^2 + z^2},$$

where $u,v,w = \Omega(u,v)$ are kept constant. If on the other hand we keep $x,y,z$ fixed and vary $u,v,w$ equation (1) represents one sheet of a hyperboloid $H$ of revolution. The surface $D$ is the envelope of all the surfaces $E$ obtained by letting $u,v,w$ vary on $S$, and vice versa $S$ is the envelope of all surfaces $H$ obtained by letting $x,y,z$ vary on $D$.

We have shown [2] how this remark can be used to determine graphically points of $D$ together with their tangent planes from a finite number of points of $S$. Our present problem is concerned with the position of the recording instruments. We shall show that the data gained from certain critical curves may be insufficient for the determination of the transforms of these curves under $T$. In practice the instruments will be distributed over a whole area around $O$, and hence these critical curves are mostly of no practical importance. It is an amusing geometrical problem, however, to give a complete characterisation of them; they turn out to be arcs on certain hyperbolas.

## 2. *Transformation of Curves*

We first turn to the following question: What information about $D$ can we obtain if we measure the reflection times along a certain curve? In orther words, assuming that we know a curve $K$ on $S$ which surfaces are then possible discontinuity surfaces?

Three close points of $K$ are transformed by $T$ into three close ellipsoids $E$, and their common point (or points) tend to a limiting position which will be a point of a curve $k$, image of $K$ under $T$. All possible surfaces $D$ must contain $k$.

We find $k$ analytically according to the theory of envelopes by substituting the three functions $u(t), v(t), w(t)$ representing $K$ in (1) and differentiating twice with respect to $t$. It is more convenient first to rewrite (1) in terms of the quantity $r = \sqrt{x^2 + y^2 + z^2}$. This yields the equation

$$ux + vy - wr - \tfrac{1}{2}(u^2 + v^2 + w^2) = 0 \tag{2}$$

It is linear in the new coordinates $x, y, r$, and slightly more general than (1). The latter equation defines a real surface $E$ only for $u^2 + v^2 \leq w^2$, while (2) represents for $u^2 + v^2 > w^2$ hyperboloids or planes.

We call the separating cone $u^2 + v^2 = w^2$ the critical cone. Our surface $S$ is situated entirely in its interior. From (1) we also conclude that the tangent planes of $H$ (which coincide with those of $S$) are never inclined by more than 45° towards the $u, v$ plane. Hence no curve on $S$ makes an angle greater than 45° with the $u, v$ plane.

Differentiating now (2) with respect to $t$ we obtain the following linear system for $x(t), y(t), r(t)$ defining the curve $k$.

$$\begin{cases} u(t)\, x + v(t)\, y - w(t)\, r = \tfrac{1}{2}(u^2 + v^2 - w^2) \\ u'\, x + v'\, y - w'\, r = \tfrac{1}{2}(u^2 + v^2 - w^2)' \\ u''\, x + v''\, y - w''\, r = \tfrac{1}{2}(u^2 + v^2 - w^2)''. \end{cases} \tag{3}$$

## 3. *Definition of Critical Curves.*

The solutions os system (3) remain undetermined if its augmented matrix

$$\begin{pmatrix} u, & v, & -w, & \tfrac{1}{2}(u^2 + v^2 - w^2) \\ u', & v', & -w', & \tfrac{1}{2}(u^2 + v^2 - w^2)' \\ u'', & v'', & -w'', & \tfrac{1}{2}(u^2 + v^2 - w^2)'' \end{pmatrix} \tag{4}$$

has a rank less than 3. A curve for which this condition holds in every point will be called critical.

The determination of the image of a critical curve under $T$ requires higher derivatives than the second. It is therefore not advisable to place the seismometers on the projection onto the $u, v$ plane of a critical curve because we might by chance pick up by our measurements just such a critical curve.

*4. Determination of all critical Curves.*

The four determinants of order 3 in matrix (4) are all Wronskian determinants. Their identical vanishing entails the existence of a constant matrix $(\lambda_{ik})$ each of whose rows contains a non-zero vector such that the following four equations hold identically in $t$.

$$\lambda_{01}u + \lambda_{02}v + \lambda_{03}w = 0 \tag{5}$$

$$\lambda_{11}u + \lambda_{12}v + \lambda_{13}(w^2 - \zeta^2) = 0 \tag{6}$$

$$\lambda_{21}u + \lambda_{22}w + \lambda_{22}(w^2 - \zeta^2) = 0 \tag{7}$$

$$\lambda_{31}v + \lambda_{32}w + \lambda_{33}(w^2 - \zeta^2) = 0, \tag{8}$$

where $\zeta = \sqrt{u^2 + v^2}$ .

We observe that every straight line passing through the origin and every straight line making an angle of 45° with the $u,v$ plane satisfy these conditions. Hence they are critical.

Since each of the equations (5-8) represents either a plane or a surface of the second degree passing through $O$ all the other critical curves are conics passing through $O$. Let us first assume that the plane of such a conic is not perpendicular to the $u,v$-plane (all the results remain valid in the latter case however). This plane is then given by an equation

$$w = \lambda_1 u + \lambda_2 v, \tag{9}$$

and the projection of the conic on the $u,v$-plane by

$$au^2 + buv + cv^2 + du + ev = 0. \tag{10}$$

Substitution of (9) in (6) gives a relation between $u$ and $v$ which must be the same as (10). Hence a constant $\Lambda \neq 0$ must exist such that

$$\Lambda\lambda_{11} = d, \qquad \Lambda\lambda_{12} = e,$$

$$\Lambda\lambda_{13}(\lambda_1^2 - 1) = a, \quad 2\Lambda\lambda_{13}\lambda_1\lambda_2 = b, \quad \Lambda\lambda_{13}(\lambda_2^2 - 1) = c.$$

Since we ruled out the straight lines at least one of the numbers $a$, $b$, $c$ does not vanish. Therefore, $\lambda_{13} \neq 0$. Hence the last three equations yield

$$a = f(\lambda_1^2 - 1), \quad b = 2f\lambda_1\lambda_2, \quad c = f(\lambda_2^2 - 1),$$

where $f$ is a non-vanishing constant. The same solutions are obtained by intersection of (9) and (7, 8). It is easy to see that these necessary conditions are also sufficient, and our critical conics can therefore be represented by

$$w = \lambda_1 u + \lambda_2 v, \quad (\lambda_1^2 - 1)u^2 + 2\lambda_1\lambda_2 uv + (\lambda_2^2 - 1)v^2 + Du + Ev = 0,$$

where $\lambda_1$, $\lambda_2$, $D$, $E$ are arbitrary parameters.

Some elementary analytical geometry leads from this expression to the following complete geometrical characterization of the critical curves: The set of critical curves consists of

a) the straight lines passing through the origin,

b) the straight lines making an angle of 45° with the $u,v$-plane,

c) the ellipses passing through $O$, whose minor axes are parallel to the $u,v$-plane and between whose eccentricity $\epsilon$ and whose angle $\omega$ of their plane with the $u,v$-plane, the following relation holds:

(11) $$\epsilon = \sqrt{2}\,\sin\omega,$$

d) the parabolas passing through $O$ whose axis and whose planes make angles of 45° with the $u,v$-plane,

e) the hyperbolas passing through $O$ whose transverse axes are parallel to the $u,v$-plane and for which (11) also holds.

*5. Projection of the critical curves on the u,v-plane.*

For the projections of these critical curves we find the following properties:

Every conic $C'$ of the $u,v$-plane passing through $O$ is the projection of a critical curve. Its plane is found by the following rules: Its intersection with the $u,v$-plane passes through $O$ and is

a) parallel to the minor axis of $C'$ if $C'$ is an ellipse;

b) parallel to the transverse axis of $C'$ if $C'$ is a hyperbola;

c) perpendicular to the axis of $C'$ if $C'$ is a parabola.

In all these three cases its angle with the $u,v$-plane is found from

(12) $$\epsilon' = \operatorname{tang}\omega,$$

where $\epsilon'$ is the eccentricity of $C'$.

Not the whole of these conics, however, is "dangerous", there are only certain arcs which should be avoided. It is clear that those arcs of a critical curve which are in the exterior of the critical cone $w^2 = u^2 + v^2$, and those arcs making angles greater than 45° with the $u,v$-plane, can never be obtained by actual measurements, since their images on $D$ are imaginary.

"Forbidden" are therefore only those arcs of a conic $C'$ for which the corresponding arcs of $C$ are in the interior of the critical cone and which at the same time make angles less than 45° with the $u,v$-plane

According to (11) the planes of critical ellipses and parabolas are either outside or tangent to the critical cone. Hence no arc of an ellipse or parabola is forbidden.

The plane of a hyperbola $C$ however intersects the critical cone in two real generating lines $g_1$, $g_2$ which make an angle $\beta$ given by

$$\tan \beta = \sqrt{-\cos 2w}\ . \tag{13}$$

From (11) the angle between the asymptotes of $C$ can be computed, and comparing with (13) it is found that $\beta = \alpha$. Using (12) and the fact that the transverse axis of $C$ is parallel to the $u,v$-plane, it can easily be proved that this relation is preserved in the projection. Thus $\beta' = \alpha'$, $2\beta'$ being the angle between the projections $g_1'$, $g_2'$ of $g_1$, $g_2$ and $2\alpha'$ being the angle between the asymptotes of $C'$. Hence, if the two perpendicular lines $g_1'$, $g_2'$ are drawn from $O$ to the asymptotes of $C'$, they define four angular sectors; the arcs $C_1'$ of $C'$ included in those two sectors whose bissector is perpendicular to the transverse axis of $C'$ are projections of arcs of $C$ situated within the critical cone.

Let us determine now those arcs $C_2$ of $C$ whose tangents make angles with the $u,v$-plane less than 45°.

Since the transverse axis of $C$ is parallel to the $u,v$-plane there are four of these arcs, situated symmetrically with respect to both axes and extending to infinity. We determine their boundary points in the following way:

If a line-element is situated in a plane making an angle $\omega$ with the $u,v$-plane and if its projection includes an angle $\chi$ with the trace of its plane then its slope is determined by

$$\text{tang}\ \tau = \sin \chi\ \text{tang}\ \omega.$$

From $\tau \leq 45°$ we conclude, using (12), that

$$\sin \chi \leq 1/\epsilon' = \cos \alpha',$$

where $\alpha'$ is the acute angle between an asymptote and the transverse axis of $C'$. Hence the limit points of the dangerous arcs of $C'$ are characterized by

(14) $$\chi_o = 90^\circ - \alpha',$$

where $\chi_o$ is the angle of the tangent of $C'$ with its transverse axis. This equation expresses the fact that the tangents in the boundary points are perpendicular to the asymptotes, and therefore parallel to the above mentioned lines $g_1'$, $g_2'$.

Since the slope of a tangent of a hyperbola is numerically greater than the slope of the asymptotes we have always

(15) $$\chi > \alpha'.$$

Hence if $\alpha' \geq 45^\circ$ there is no value $\chi_o$ which satisfies simultaneously (14) and (15) and therefore there are no forbidden arcs on the hyperbola $C'$.

If $\alpha' < 45^\circ$, however, it is always possible to draw 4 tangents $t_i$ to $C'$ perpendicular to the asymptotes. Their points of contact $A$, $B$, $C$, $D$ are the boundary points of those 4 arcs $C_2'$ of $C'$ which we are looking for.

Summing up, we may state:

a) all straight lines in the $u,v$-plane passing through $O$ are forbidden;

b) every hyperbola passing through $O$ whose asymptotes include an angle $2\alpha'$ less than $90°$ (measured in that sector which contains the curve) contains one or two forbidden arcs. They are the common parts of the arcs $C_1'$ and $C_2'$ determined above.

Given an arbitrary hyperbola $C'$ with asymptotes making an angle less than $90°$ we can convice ourselves of the truth of the above statement by letting $O$ vary along $C'$. The points $A$, $B$, $C$, $D$ are independent of the position of $O$. Let $E$ be the intersection of the parallel line to $t_1$ through $B$ with $C'$. If $C'$ is given by $\xi = a \cosh t$, $\eta = b \sinh t$, the parameter values of $A$ and $E$ are found from $\tanh t_A = -b^2/a^2$ and $t_E = 3t_A$. Our figure illustrates the case when $O$ is situated between $A$ and $E$. Other cases to be considered are: $O$ between $A$ and $B$, $O = A$, and $O$ outside of $BE$.

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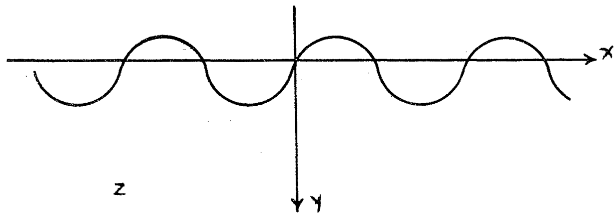


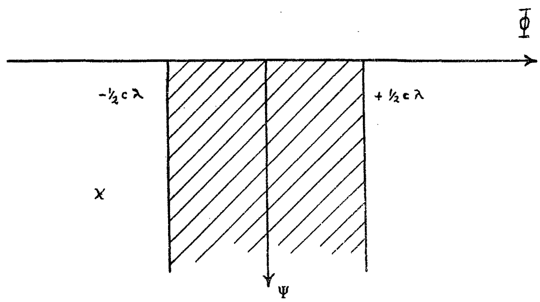


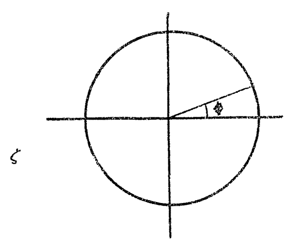


University of Alberta

*Dear Mr. James:*

"The paper presented is the work of my twelve year old boy who took more than a year to gather courage to submit it for editorial scrutiny. You may be interested to know that when my son first received a subscription to *Mathematics Magazine* about two and a half years ago, he was aghast to note that he couldn't understand a *single thing* in it. With each successive issue, however, his understanding unfolded (he is a self-taught mathematician) until now he awaits each issue with eagerness, and recently was able to submit his solution to one of the Proposals published in your last issue. He considers his subscription to *Mathematics Magazine* one of the finest presents he ever received!"

*Sincerely yours,*

*Sylvia Bergman*


What did I tell you about dividing by zero?

# SOLVING DIFFERENTIAL EQUATIONS WITHOUT COMPLEX NUMBERS

Michael J. Pascual

The only place in a course in real differential equations that complex numbers arise (embarrassingly, I'm afraid) is in the study of the linear equation with constant coefficients. This seems to imply that the theory is not complete in itself, without trespassing into the complex number field. The purpose of this paper is to show that one need not introduce complex quantities at all to obtain the solution even when the auxiliary equation does not have real roots.

The equation we are concerned with is

$$y'' + py' + qy = 0$$

where $p$ and $q$ are real constants. For convenience this can be put into the form

(1) $$y'' - 2ay' + (a^2 \pm b^2)y = 0$$

where $a = -\dfrac{p}{2}$ and $b = \sqrt{|q - (p^2/4)|}$. We first solve (1) when $a = 0$ by setting $y' = v$, $y'' = v\dfrac{dv}{dy}$ so that (1) becomes

(2) $$v\frac{dv}{dy} \pm b^2 y = 0$$

which is easily solved and yields solutions

$$v = \pm\sqrt{c \pm b^2 y^2}$$

the double sign inside corresponding to the one in (2). The solutions for $y$ are

$$y = A \sin(B \pm bx) \quad \text{and} \quad ln(by + \sqrt{B + b^2y^2}) = \pm bx + A$$

where $A$ and $B$ are arbitrary; the first solution corresponding to the minus sign of (2) and the second solution corresponding to the plus sign. With a little manipulation these can be put into the form

(3) $$y = c_1 \cos bx + c_2 \sin bx$$

(4) $$y = c_1 e^{bx} + c\ e^{-bx}$$

in which the ambigious sign is taken care of by the arbitrary character of $c_1$ and $c_2$.

Now we are ready for the general case with $a \neq 0$ and $b \neq 0$. The change of variable $y = ue^{ax}$ transforms (1) into

$$u'' \pm b^2 u = 0, \tag{5}$$

which is the same as (2): hence

$$u = c_1 \cos bx + c_2 \sin bx$$

or

$$u = c_1 e^{bx} + c_2 e^{-bx}$$

so that

$$y = e^{ax}(c_1 \cos bx + c_2 \sin bx) \tag{6}$$

$$y = e^{ax}(c_1 e^{bx} + ce^{-bx}). \tag{7}$$

where (6) holds for the plus sign in (5) and (7) holds for the minus sign.

Finally for the only case we have not covered, namely when $b = 0$ we solve (5) easily to obtain

$$u = c_1 x + c_2$$

so that

$$y = e^{ax}(c_1 x + c_2) \tag{8}$$

Now that all possibilities have been covered in a perfectly natural manner without introducing complex quantities, one might for expediency show that by solving the auxiliary equation

$$m^2 - 2am + (a^2 \pm b^2) = 0$$

and "examining" the roots we can immediately set down the solutions (3), (4), (6), (7), or (8) depending upon whether the roots are pure imaginary, real and negatives of one another, complex conjugates, real and unequal in absolute value, or real and equal respectively. Of course once (1) has been disposed of then, by the usual development of operators in factored form, higher ordered equations could be solved.

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# SOLUTION OF TRIANGLES ON THE SLIDE RULE

V. C. Harris

The following examples show how the law of sines may be used with the slide rule to solve triangles in case three sides, or two sides and the included angle, are given. Methods of solution by use of the law of sines in other cases are well known. Hence, subject to the limitations of the slide rule, the law of sines suffices to solve any triangle.

The method involves successive approximations. The adjustment made at each step is estimated on the basis of the difference between 180° and the sum of the angles obtained by the previous approximation. Skill can be developed rapidly and the triangle usually solved in three or four settings.

We take up first the case of three sides given.

*Example 1.* Given $a = 4.8$; $b = 7.3$ and $c = 8.2$. Opposite the largest side (8.2) on the $D$ scale set 90° on the $S$ scale. (We will call the scales $D$ and $S$ hereafter and these will be the only scales with which we shall be involved.) The angles on $S$ opposite the three given sides on $D$ and their total are:

| | |
|---|---|
| $C$ : | 90° |
| $B$ : | 63° |
| $A$ : | 35.9° |
| Total | 188.9° |

Since the total is not 180°, the angles obtained do not yield a solution. The triangle is not a right triangle since if it were, the angles just found would have to be the solution. Moreover the triangle must be acute since the sum is more than 180° and if $C$ were increased, both $A$ and $B$ would be decreased but (as the slide rule shows) by an amount less than the increase in $C$ and the total of the angles would always be too large. We now try to reduce the sum of the angles to 180°. If we should take 8.9° off angle $C$ so that opposite 8.2 on $D$ we would have 81.1° on $S$, then we would also be taking off of angles $A$ and $B$ and our result would look like this:

| | |
|---|---|
| $C$ : | 81.1° |
| $B$ : | 61.7° |
| $A$ : | 35.3° |
| Total | 178.1° |

We would have compensated too much and wish instead to allow for this. We should take off *less* than the discrepancy and we estimate that we should take off about 8° instead of 8.9°. The result is thus:

| | |
|---|---|
| $C$ : | 82 ° |
| $B$ : | 61.8° |
| $A$ : | 35.4° |
| Total | 179.2° |

We have still taken off too much, but we see that changing $C$ by 8° took off only 1.2° from $B$ and .5° from $A$. Hence most of the .8° which the total of the angles needs to make 180° will come from increasing $C$ . If we try 82.6° we find

| | |
|---|---|
| $C$ : | 82.6° |
| $B$ : | 61.9° |
| $A$ : | 35.5° |
| Total | 180.0° |

Hence these values give the solution.

The work may be put down compactly as follows:

| | | First setting | Second setting | Third setting |
|---|---|---|---|---|
| $c = 8.2$ | $C$ : | 90 | 82 | 82.6 |
| $b = 7.3$ | $B$ : | 63 | 61.8 | 61.9 |
| $a = 4.8$ | $A$ : | 35.9 | 35.4 | 35.5 |
| | | 188.9 | 179.2 | 180.0 |

In case the largest side does not start with the largest digit, it is convenient to multiply or divide the sides by the same integer to have the largest side have the largest initial digit. The purpose of this step is to make all readings occur to the *left* of the index of $S$ so that the slide need not be reversed. Also, if the sum of the angles is less than 180° when the first setting is made, then the largest angle of the triangle is obtuse, and the supplement of the angle on $S$ must be used in evaluating that largest angle.

*Example* 2. Given $a = 131$; $b = 77$ and $c = 92$. Divide the sides by 2 and solve the similar triangle as indicated:

| | | First setting | Second setting | Third setting | Fourth setting |
|---|---|---|---|---|---|
| $a = 65.5$ | $A$ : | 90 | 100 | 101 | 101.4 |
| $b = 38.5$ | $B$ : | 36 | 35.4 | 35.2 | 35.1 |
| $c = 46$ | $C$ : | 44.6 | 43.8 | 43.6 | 43.5 |
| | | 170.6 | 179.2 | 179.8 | 180.0 |

The angles obtained in the fourth setting are the solution. When the slide is moved to the right, angles $B$ and $C$ are diminished in the present example. Hence $A$ must be increased to obtain a total of 180°. The largest angle must be obtuse and is the supplement of the angle read on $S$. Note that in increasing $A$, since angles $B$ and $C$ decrease, we must increase $A$ by *more* than the discrepancy in each trial. For example, the discrepancy at the first setting in 9.4°, so we estimated an adjustment of 10° which we then applied, but this turned out to be too small.

When two sides and the included angle are given, the procedure is much the same. Again, if the larger side does not start with the larger digit, it is convenient to solve a similar triangle first, and then multiply or divide by the proper number to get the unknown side.

*Example 3.* Given $b = .91$; $c = 1.25$ and $A = 71°$. First, multiply the sides by 5 and solve the similar triangle $b = 4.55$; $c = 6.25$ and $A = 71°$. The work follows:

| | | First setting | Second setting | Third setting | Fourth setting |
|---|---|---|---|---|---|
| $a =$ | $A$ : | 71 | 71 | 71 | 71 |
| $b =$ 4.55 | $B$ : | 46.8 | 40.5 | 42.4 | 42 |
| $c =$ 6.25 | $C$ : | 90 | 63 | 68 | 67 |
| | | 207.8 | 174.5 | 181.4 | 180.0 |

Side $a$ is read on $D$ opposite 71° on $S$ as 6.42, so in the original triangle we have $a = 1.28$. Note that we still let the largest unknown angle be 90° for a first approximation. In using 63° for $C$ in the second approximation we show that not very much skill was being employed since almost no allowance was made for the expected decrease in $B$. Consequently the total change in the sum of the angles turned out to be 5.5° too much.

In a few instances, some angles must be read on the $ST$ scale. Also, the number of settings needed varies. However, the preceding remarks and a little experience will show the user what happens to the small angle or angles as the largest one is adjusted. This enables one to vary the adjustment of the large angle in the correct direction and cuts down on the number of settings needed.

When compared to alternative methods, the above method seems simple in idea and rapid in practice. It also has the advantage of using the same trigonometric law as that commonly employed in solving triangles under the other cases, a law which can be explained to students with little mathematical background.

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# A NUMBER SYSTEM WITH AN IRRATIONAL BASE

George Bergman

The reader is probably familiar with the binary system and the decimal system and probably understands the basis for any others of that type, such as the trinary or duodecimal. However, I have developed a system that is based, not on an integer, or even a rational number, but on the irrational number $\tau$ (tau), otherwise known as the "golden section", approximately 1.618033989 in value, and equal to $(1+\sqrt{5})/2$.

In order to understand this system, one must comprehend two peculiarities of the number $\tau$. They are based on tau's distincive property [1] that

$$\tau^n = \tau^{n-1} + \tau^{n-2}$$

(a.) Take any approximation ($A_1$) of $\tau$. Taking the reciprocal, we get a number ($a_1$) that is proportionately the same distance from $1/\tau$ as $A_1$ was from $\tau$, but arithmetically nearer. Adding 1,[2] we get a number ($A_2$) that is proportionately nearer $\tau$ than $a_1$ was to $1/\tau$ but arithmetically just as near. Since $a_1$ is arithmetically nearer than $A_1$, $A_2$ is nearer in both respects to $\tau$ than $A_1$. Repeating the process of taking the reciprocal and adding 1, we approach $\tau$. Now, taking 1 as $A_1$, and expressing our approximations of $\tau$ (i.e. $A_1, A_2, A_3$, etc.) as fractions, we get

$$\frac{1}{1}\ \frac{2}{1}\ \frac{3}{2}\ \frac{5}{3}\ \frac{8}{5}\ \frac{13}{8}\ \ldots\ldots\ldots$$

Taking either the numerators or the denominators, we get what is know as the Fibonacci Series, *each term of which is formed by adding the two previous terms*[3]; for

$$\frac{f_{n+2}}{f_{n+1}} = 1 + \frac{1}{\left(\frac{f_{n+1}}{f_n}\right)} = 1 + \frac{f_n}{f_{n+1}} = \frac{f_{n+1}+f_n}{f_{n+1}} \quad \text{so } f_{n+2} = f_{n+1} + f_n$$

(We designate the $n^{\text{th}}$ term of the Fibonacci Series by $f_n$, setting $f_1 = 1$, $f_2 = 1$. This practice shall be used throughout the article.)

(b.) Any integral power of $\tau$ can be expressed in the form $\tau^n = A\tau + B$, where $A$ and $B$ are integers and, in fact, numbers in the Fibonacci Series. The explanation of this startling fact is really rather simple:

Since

$$\tau^1 = 1\tau + 0 \quad \text{and} \quad \tau^2 = 1\tau + 1,$$

and since

$$\tau^3 = \tau^2 + \tau^1; \qquad \tau^3 = (1\tau + 0) + (1\tau + 1) = 2\tau + 1.$$

In the same way

$$\tau^4 = \tau^2 + \tau^3 = (1\tau + 1) + (2\tau + 1) = 3\tau + 2.$$

In general

$$\tau^n = \tau^{n-1} + \tau^{n-2} = (f_{n-1}\tau + f_{n-2}) + (f_{n-2}\tau + f_{n-3})$$

$$= (f_{n-1} + f_{n-2})\tau + (f_{n-2} + f_{n-3})$$

$$= f_n\tau + f_{n-1} \qquad ^4 \text{ (see Note 2.)}$$

Can this be applied to negative powers of $\tau$? We don't know any Fibonacci numbers before 1, but it is easy to see how we can find them: Taking 1 and 1 as our first two, we can see that the term before must be 0, since that is the only number which, when added to 1 gives 1. In the same way, the number before that must be 1, since 1 is the only number that, when added to 0 gives 1; and the next term must be $-1$, since no other number gives 0 when added to 1. Continuing this process, we get 0, 1, $-1$, 2, $-3$, 5, $-8$, 13, $-21$... . Obviously, this is alternately $+1$ and $-1$ times the corresponding Fibonacci numbers. But can this be proved to be true in all cases? It can by induction. The rule we want to prove, expressed as an equation, is:

$$f_{-y} = (-1)^{y+1} f_y$$

Let us assume it true for $y = 1, 2, \ldots n$. Now by the basic property of the Fibonacci Series:

$$f_{-n} + f_{-n-1} = f_{-n+1}$$

$$f_{-n-1} = f_{-n+1} \quad f_{-n}$$

$$f_{-(n+1)} = (-1)^n f_{n-1} - (-1)^{n-1} f_n$$

$$f_{-(n+1)} = (-1)^{n+2} (f_{n-1} + f_n)$$

$$f_{-(n+1)} = (-1)^{n+2} f_{n+1}$$

The inductive proof is completed by the examples already cited.

Applying this to powers of $\tau$, we make a list of them from $\tau^{-5}$ to $\tau^5$:

| | |
|---|---|
| $r^{-5} = 5r - 8$ | $r^{0} = 0r + 1$ |
| $r^{-4} = -3r + 5$ | $r^{1} = 1r + 0$ |
| $r^{-3} = 2r - 3$ | $r^{2} = 1r + 1$ |
| $r^{-2} = -1r + 2$ | $r^{3} = 2r + 1$ |
| $r^{-1} = 1r - 1$ | $r^{4} = 3r + 2$ |
| | $r^{5} = 5r + 3$ |

Now, at last, we shall get back to our concept of a system based on $r$. Like the binary system, it can have only two symbols: 1 and 0. But, unlike the binary system, it has the rule (2): 100 = 011 [5] (place the decimal point anywhere – it's a general rule). But how do we find the numbers? We know that 1 is $r^0$ or 1.0. Next, looking at the table of powers of $r$, one notices that

$$r^1 = 1r + 0 \quad \text{and} \quad r^{-2} = -1r + 2.$$

Adding them together, one gets

$$r^1 + r^{-2} = (1r + 0) + (-1r + 2) = 2.$$

Therefore, 2 = 10.01 (in this system). Of course, because of rule (2), this can also be expressed as 1.11, [6] 10.0011, 10.001011, 1.101011, etc., but 10.01 is what I call the simplest form (that form in which there are no two 1's in succession, and which, therefore, cannot be acted upon by the reverse of rule (2), called simplification (11 = 100). To convert a number to its simplest form, repeatedly simplify the leftmost pair of consecutive 1's.

To continue with our "translation" of numbers into this system, we notice (after a careful examination of the table) that $r^2 = 1r + 1$ and $r^{-2} = -1r + 2$, and adding them together $r^2 + r^{-2} = 3$, and so 3 is 100.01 in this system. What about 4? Well, since $r^2 + r^{-2} = 3$, $r^2 + r^{-2} + r^0$ (101.01) must equal 4, since $r^0 = 1$. Can this method of adding 1 be used for other numbers? The answer is "yes"; just convert the number into the form in which there is a zero in the units column and place a 1 in it. If the method of conversion is not obvious, use this method:

a. Change to the simplest form.

b. If there is no 1 in the units column, you are finished. If there is, look in the $r^{-2}$ column (there can't be any in the $r^{-1}$ column because it is in its simplest form and there is a 1 in the column next to it).; if there is a 0 there, expand [7] the 1 into the $r^{-1}$ and $r^{-2}$ columns (that's all); if there is 1, look in the $r^{-4}$ column; if there is a zero there, expand the 1 in the $r^{-2}$ column into the $r^{-3}$ and $r^{-4}$ column and the 1 in the units column into the $r^{-1}$ and $r^{-2}$ columns. If there is a 1, look in the $r^{-6}$ column; if there is a zero, expand the 1 in the $r^{-4}$ column into the $r^{-5}$ and $r^{-6}$ columns, the 1 in the $r^{-2}$

column into the $r^{-3}$ and $r^{-4}$ column, and the 1 in the units column into the $r^{-1}$ and $r^{-2}$ columns; if on the other hand there is a 1, look in the $r^{-8}$ column, etc. If it is the endless fraction 1.0101010 1..., change it to 10.000000 ... .

We can now construct a table of integers in this system. Here are those from 0 to 14 (in their simplest forms)

| | | |
|---|---|---|
| 0 – 0 | 5 – 1000.1001 | 10 – 10100.0101 |
| 1 – 1 | 6 – 1010.0001 | 11 – 10101.0101 |
| 2 – 10.01 | 7 – 10000.0001 | 12 – 100000.101001 |
| 3 – 100.01 | 8 – 10001.0001 | 13 – 100010.001001 |
| 4 – 101.01 | 9 – 10010.0101 | 14 – 100100.110110 |

*Examples of the basic processes*

1. Change 100101.111001 (equals 16) to the simplest form. The first pair (farthest left) is in the units and $r^{-1}$ column, so we simplify it into the $r^{1}$ column, giving 100110.011001. This time the pair farthest to the left is the one we have just created with our new 1 in the $r^{1}$ column (the result of our simplification), added to the 1 already in the $r^{2}$ column. This we simplify into a 1 in the $r^{3}$ column, which gives us 101000.011001. Finally, we change the last remaining pair (in the $r^{-2}$ and $r^{-3}$ columns) into a 1 in the $r^{-1}$ column, arriving at our final answer: 101000.100001

2. Change 101.01 (4) to a form with a zero in the units column. (i.e. a form to which 1 can be added). One can see that it is already in its simplest form. However, there is a 1 in the units column, and we must remove it. The first thing we do is look in the $r^{-2}$ column; since there is a 1 there, we look in the $r^{-4}$ column. This is empty, and so we expand the 1 in the $r^{-2}$ column into the $r^{-3}$ and $r^{-4}$ columns, getting 101.0011. Now that the $r^{-2}$ column is empty, we can expand the unit into a pair in the $r^{-2}$ and $r^{-1}$ columns, getting 100.1111, which has a zero in the units column. We can now add 1 to it:

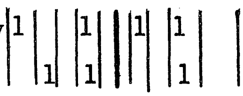
100.1111+1 = 101.1111 = 110.0111 = 1000.0111 = 1000.1001 (five)

*The Arithmetic Operations*

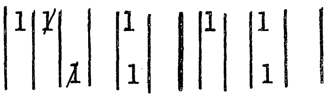
The arithmetical operations, although they are basically the same as in any other system, are, in practice, quite different because of the peculiarities of this system. As our first step in all of them, we eliminate zeros, which would only hinder us, and show the place values of 1's by actural placement in columns. For instance, four (101.01) would be |1| |1‖ |1|, the heavy line representing the "decimal point". The necessity for this step results from the fact that, though in the systems to which we are accustomed, the steps of addition are simple enough to be performed mentally, this is not so in the tau system; nor is each column non-dependent on the one to the left of it. It is thus necessary to have lined columns in which to carry out the work.

Now for the actual processes, we shall start with addition. The example

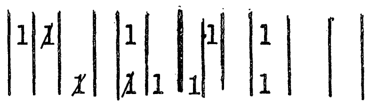
$$\begin{array}{r} 10010.0101 \\ +\ 1010.0001 \end{array} \begin{pmatrix} 9 \\ +6 \end{pmatrix}$$ would be represented by



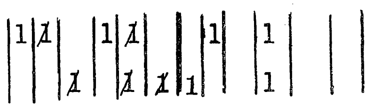
In this set-up it can be seen that we have a pair, consisting of a 1 in the $\tau^3$ column and one in the $\tau^4$ column. This we simplify into a 1 in the $\tau^5$ column.



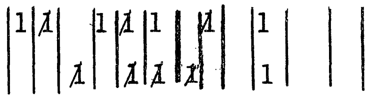
Now, however, we have no obvious way to continue. We are left with two 1's in the same column. We can neither add them together to give 2 (as we would in the decimal system), nor is there any simple "carrying" operation. We must, therefore, change this to a form not having two 1's in the same column. We will start by expanding one of the 1's in the $\tau^1$ column:



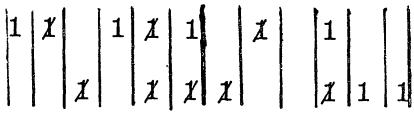
Now we can simplify the pair we have just created in the $\tau^1$ and units columns:



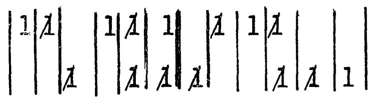
and the one in the $\tau^{-1}$ and $\tau^{-2}$ columns:



We shall now use the same type procedure for the 1's in the $\tau^{-4}$ column. We expand one of the 1's there:



and simplify in the $\tau^{-4}$ and $\tau^{-5}$ columns:



and express our answer in ordinary form, writing 1's in columns with an un-crossed-out 1 and 0's in the columns where all have been crossed out:

100101.001001 (15)

For general rules as to procedure, I believe that these will do in most cases. (These general rules and the ones for the other processes are not the types of rules that, if disobeyed, would give the wrong answer, but merely guides to the quickest way to get the right one):

a) Expand only when that is the only way to remove a 1 from the same column as another, regardless of whether this will result in the same situation in another column, but only if no more simplification can be done.

b) Simplify whenever possible, and, if there are two or more pairs, always simplify the one farthest to the left first. Because of this rule, simplification should never result in two 1's in the same column, i.e. |1|1|1| should be simplified into |1|~~1~~|~~1~~|1| rather than

| | | | |
|---|---|---|---|
| 1 | | | |
| 1 | ~~1~~ | ~~1~~ | |

*Subtraction*

Subtraction is the next process I shall describe. As in addition, we set up the numbers in columns, but here we shall assign negative values to the 1's from the subtrahend. For instance, to find (11–6) we set up

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | 1 | | 1 | 1 | 1 |
| | -1 | | -1 | | | -1 |

We now "cancel" the 1 and the -1 in the $r^{-4}$ column, giving

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | 1 | | 1 | 1 | ~~1~~ |
| | -1 | | -1 | | | -~~1~~ |

Next, we expand the 1 in the $r^2$ column, getting

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | ~~1~~ | 1 | 1 | 1 | ~~1~~ |
| | -1 | | -1 | 1 | | -~~1~~ |

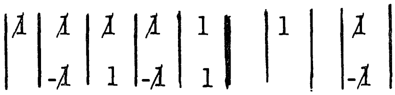
Again, we cancel, this time in the $r^1$ column,

| | | | | | | |
|---|---|---|---|---|---|---|
| 1 | | ~~1~~ | ~~1~~ | 1 | 1 | ~~1~~ |
| | -1 | | -~~1~~ | 1 | | -~~1~~ |

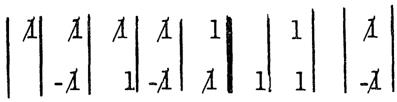
And again we expand, this time the 1 in the $r^4$ column, getting

| | | | | | | |
|---|---|---|---|---|---|---|
| ~~1~~ | 1 | ~~1~~ | ~~1~~ | 1 | 1 | ~~1~~ |
| | -1 | 1 | -~~1~~ | 1 | | -~~1~~ |

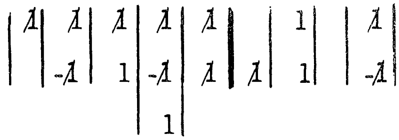
For the third time we cancel, (in the $r^3$ column) giving



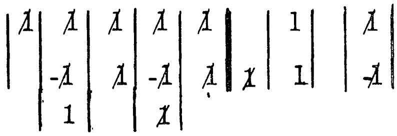
This we treat just as we would if we were adding and arrived at this stage; by expanding one of the 1's in the units column we get



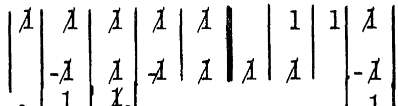
Next we simplify the pair in the units and $r^{-1}$ columns getting



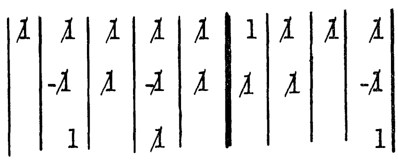
and the pair we thereby form in the $r^1$ and $r^2$ column getting



Finally, we expand one of the 1's in the $r^{-2}$ column, getting



and simplify the resulting pair (in the $r^{-2}$ and $r^{-3}$ columns) getting as our final answer



or 1000.1001 (5)

For subtraction it is harder to formulate a general rule, but I think it would suffice to say: Cancel whenever possible and simplify or expand whenever that would permit cancellation (also remember not to confuse a 1 with a -1 (|•|1| -1| ≠ |1| $\not 1$| -$\not 1$|) and not to make the mistake of "expanding" a -1 into two +1's.) After all -1's have been removed by cancellation, proceed as you would with an addition example.

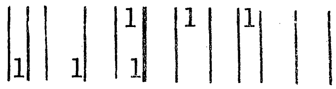
*Multiplication*

Multiplication involves nothing new. We simply place the partial products as we do in the decimal system, and add. For instance:

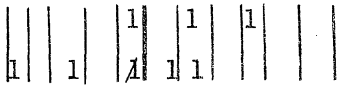
```
   101.01
 × 100.01   (3 × 4)
```

Setting up the partial products, we get

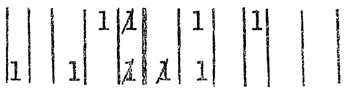
```
  10101  or
10101
```



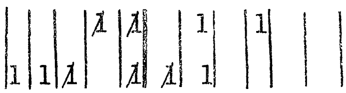
and (from now on it is simple addition) expanding one of the 1's in the units column:



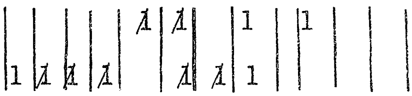
We now simplify the pair in the units and $r^{-1}$ column



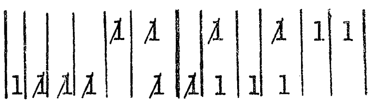
and the pair we thus produce:



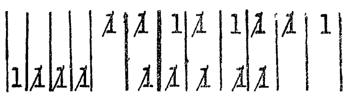
and again the pair this simplification produces, getting:



Next we expand one of the 1's in the $r^{-2}$ column, and one of the 1's in the $r^{-4}$ column:



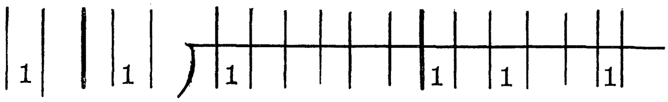
Finally, we simplify the pair in the $r^{-2}$ and $r^{-3}$ columns, and then the one in the $r^{-4}$ and $r^{-5}$ columns, giving our final answer:



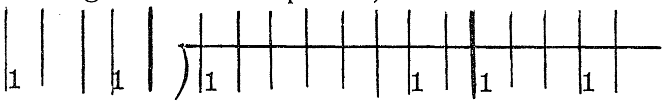
*or* 100000.101001

*Division*

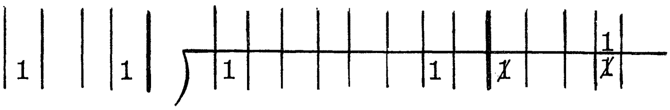
Division is quite different in this system, and is, in fact, rather odd. The only things it has in common with ordinary division are the basic principles behind it, the way the example looks, and the movement of the "decimal point" to eliminate any figures to the right of it in the divisor. It is best explained by an example: 12 divided by 2, or



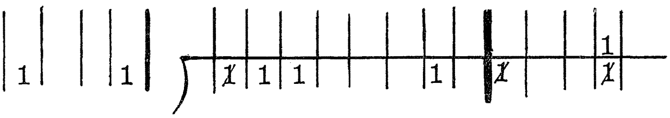
which, after moving the "decimal point", is



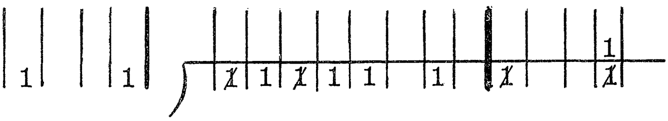
Now, since we are dividing by |1| | |1|, if there are anywhere two 1's with two spaces between them (the spaces can be empty or full), they can be crossed out and a 1 placed in the quotient above the rightmost of the two. This crossing out in no way signifies that the 1's should not be there, but is merely equivalent to, in long division (decimal) the subtraction of the product of the number placed in the quotient and the divisor from the dividend. Since the number placed in the quotient can only be 1 (placed in any column, of course), we merely subtract the dividend (placed in that same column), i.e. cross it out. It is obvious that once the whole dividend has been crossed out, the group of 1's in the quotient, after being changed to the simplest form, will be the complete quotient. Getting back to our original problem, we see that we do have just such a set of 1's in the $r^{-1}$ and $r^{-4}$ column, and so we cross it off and place a 1 in the quotient, getting



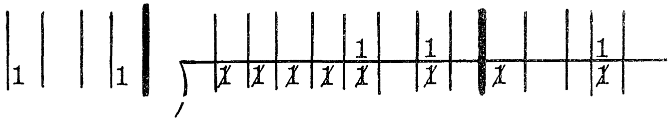
But now, you may say, there are no more pairs of 1's spaced in that way; what shall we do? The answer is our old pair of friends, expansion and simplification. Since they do not change the value of a number, if either of those processes yields a set of 1's spaced correctly, that set can be crossed off and a 1 placed in the quotient just as though that set were part of the original number. Since in our problem it is so far impossible to simplify, we shall expand. Expanding the 1 in the $r^7$ column, we get



No such set yet. However, when we expand the 1 we've just placed in the $r^5$ column, giving



we have not one but two such sets (remember that all we need is two 1's with that certain separation, regardless of intervening and crossed out 1's), one made of the 1's in the $r^1$ and $r^4$ columns, and the other of the 1's in the $r^3$ and $r^6$ columns. Crossing them both out and placing the 1's in the correct places in the quotient, we get

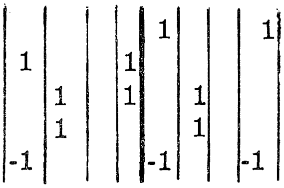


and since there are no more 1's in the dividend, our number in the quotient is the complete quotient, and so 1010.0001, or 6, is our answer. This time the general rule is: Always take that course of action that will place your next 1 (i.e., a 1 in the quotient–set in the dividend) farthest to the left. By a course of action, I mean a series of expansions and simplifications and the exchange of a set for a 1 in the quotient that follows; or simply that exchange, if the set is already there. (I did not obey this rule in my demonstration so that I could show the process in a simpler way.) This is so that the answer be in its simplest form.

By the way, the processes of addition, subtraction, and often multiplication, can be performed together by writing the addends, the subtrahends, and the partial products in one set of columns; for instance: $2\times 3+4+3-5$:

$$\begin{array}{l} 10.01 \\ \times 100.01+101.01+100.01-1000.1001 \end{array}$$

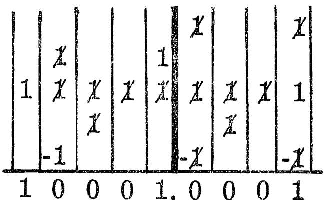
and working it out:



partial products

addends

subtrahend

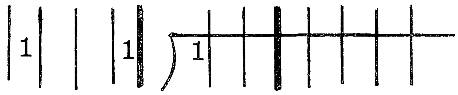


(8)

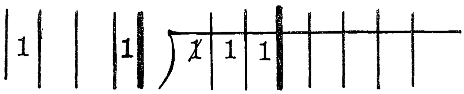
Now that we know these four processes, we have a much better way of finding a number in this system than merely repeatedly adding 1's until we reach it. For instance, to find thirty-seven, we can multiply $6\times 6$ and add 1; to check the arithmetic, we multiply $7\times 5$ and add 2:

$$\begin{array}{l} 1010.0001 \\ \times 1010.001+1 \end{array} \qquad \text{(to check)} \qquad \begin{array}{l} 1000.1001 \\ \times 10000.0001+10.01 \end{array}$$

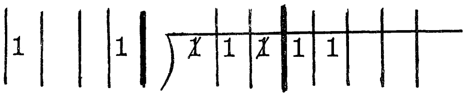
What's more, we are not only able to find integers, but, since we can divide, we ought to be able to find fractions also. Let us try. First, we shall attempt to find ½. We begin by setting up our division:



Our first set, as can be seen, will have its leftmost 1 in the $r^1$ column. We therefore expand the 1 in the $r^2$ column:



The other one we need to complete the set is a 1 in the $r^{-2}$ column; this we get by expanding the 1 in the units column:



After we exchange our set for a 1 in the quotient ($r^{-2}$ column), we notice that our remainder is 1 (in the $r^{-1}$ column). Since 1 is the number we started with, the next figure in the quotient and the next remainder should be the same as these. However, the question is, where in the quotient shall we place it? Since our first 1 was in the $r^2$ column (because we moved the decimal point) and our remainder is three places to the right of it, in the $r^{-1}$ column, our next 1 in the quotient should be three places to the right of the first 1 there. Since the next remainder will bear the same relationship to the first remainder as the first did to our original 1, the following 1 in the quotient will be three places to the right of our second 1. Since this can be carried on indefinitely, it appears that ½ expressed in the Tau System is .01001001001........ (any "doubting Thomases" may carry it out a few places to see).

Before we go on to other fractions, it would be wise to mention something about 1 in the Tau System. As you can easily see, 1=.11=.1011=.101011=.10101011 etc. It is, therefore, equal to the endless "fraction" .10101010.... (just as in the decimal system 1=.9999999....). If we can now take this fraction and expand the leftmost 1, and then expand the 1 in the $r^{-3}$ column, so as to prevent the occurrence of two 1's in the same column, and then expand the 1 in the $r^{-5}$ column so that there are not two 1's in that column, etc., we will get .01111111.... If, on the other hand, we start by expanding 1's in other columns, we get: .100111111....,

.10100111111....., 101010011111....., etc. Therefore, if you multiply .01001001001.... (½) by 10.01 (2) and get .1001111111...., this does not mean that $2 \times ½ =$ a fraction, but merely shows a different way of representing 1.

To get back to fractions, we can make a list of them just as we did of integers before:

1/2 = .010010010010...........

1/3 = .001010000010100000101000.......

1/4 = .001000000100000010000001000..........

1/5 = .00010010101001001000001001010100100100.........

1/10 = .00001000010001010000101000101010100010010100000100100010000000001...

Of course, finding these fractions is immeasurably harder than finding ½, and with 1/10 I had to work it out 5 or 10 times before I got the correct answer, as there is much room for error.

By the way, no fraction can be terminating in this system, since that would mean that it could be expressed as the sum of a group of integral powers of Tau. Since all the powers of Tau can be expressed as the sum of an integer and an integral multiple of Tau; if the integral multiples "cancel" (e.g. $\tau^{-1} + \tau^{-3} + \tau^{-4} = 1\tau - 1 + 2\tau - 3 - 3\tau + 5 = 1$) the result will be an integer, and if they don't (e.g. $2\tau - 3 - 3\tau + 5 = \tau + 2$), it will naturally be irrational. However, when we have an endless series, this paradox is detoured by admitting the fact that Lim $AT + B$ with $A$ and $B$ always integral can be a rational fraction if $A$ and $B \to \infty$.

The Tau System has a good many other interesting and unusual characteristics, and investigation by the readers of some, such as the frequency, occurrence, and nature of numbers with a 1 in the units column (when in simplest form) might prove interesting. I do not know of any useful application for systems such as this, except as a mental exercise and pastime, though it may be of some service in algebraic number theory. For instance, the numbers expressible in the Tau System in terminating form consist of all the algebraic integers in $R(\sqrt{5})$, and some of the properties of numbers in this and other systems might correspond to facts about associated fields.

*Definitions Invented for Work in the Tau System*

*Expand:* alter three successive figures of a number by changing ...100... to ...011... The result is the same in value as the original, because of rule (1). This does *not* mean change zeros to ones and ones to zeros; just this specific change.

*Simplify*: the reverse of expand; alter the figures thus: change ...011... to ...100... One speaks of simplifying the 1's in the $\tau^{n-1}$ and $\tau^{n-2}$ columns into the $\tau^{n}$ column. Also, one speaks of expanding the 1 in the $\tau^n$ column into the $\tau^{n-1}$ and $\tau^{n-2}$ columns.

*Simplest form:* that form of a number which has been simplified until no more simplification is possible. It therefore has no two 1's in succession. It also has the fewest 1's and is the easiest form to work with.

*Columns:* just as in our decimal system we speak of a units column, a ten's column, a hundred's column, a tenth's column, etc., in the Tau system we speak of a units column, a $\tau^1$ column, a $\tau^{-1}$ column, etc.

*Pair:* two 1's in succession.

*Cancellation:* a change of the form

$$\left|\begin{matrix} 1 \\ -1 \end{matrix}\right| = \left|\begin{matrix} \not{1} \\ -\not{1} \end{matrix}\right|$$

*Set:* in division, two or more 1's arranged with the same spacing as the 1's in the divisor (regardless of intervening 1's). A set can be "exchanged" for a 1 in the quotient.

[1]also true of $-1/\tau$; there are other numbers which have similar properties, e.g. there is a number $S$ between 1 and 2 for which $S^3 = S^2 + S + 1$. *Ed.*

[2]because $\tau^{-1} + \tau^0 = \tau$.

[3]this is the basic property defining the Fibonacci Series.

[4]There is also a more complex proof which involves multiplying the expressions like $2\tau+1$ by $\tau$, giving $2\tau^2+\tau$, and expanding, $\tau^2$ into $\tau+1$.

[5]This is a restatement of $\tau^n = \tau^{n-1} + \tau^{n-2}$.

[6]By changing the 1 in the $\tau$ column to 1.1.

[7]If you come across words (like "expand") used in an unfamiliar way, look for them in the list of definitions at the end of this article. I have put there all words which I have had to invent or alter for use in this system, so as not to break up the text by explaining them.

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Jr. High School 246
Brooklyn, N. Y.

(see page 91)

# PROBLEMS AND QUESTIONS

*Edited by*

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 North Vermont Ave., Los Angeles 29, California.*

## PROPOSALS

**320.** *Proposed by M. S. Krick, Albright College, Pennsylvania.*

Find three consecutive odd integers less than 10,000 which are divisible by 27, 25, and 7, respectively.

**321.** *Proposed by R. T. Coffman, Richland, Washington.*

Given a line of unit length and an acute angle $\theta$, construct with compasses and straight-edge a plane figure having an area equal to

$$\int_0^\theta \cos^2\theta \, d\theta.$$

**322.** *Proposed by D. A. Steinberg, Livermore, California.*

Show that if $x$ is any real number, $a$ any non-zero real number and $p$ any non-negative integer, then

$$e^x = \left(\frac{a+x}{a}\right)^{p+1} \left[ 1 + \sum_{j=1}^{\infty} x^j \sum_{k=0}^{j} \binom{p+k}{k} \frac{(-1)^k}{a^k (j-k)!} \right].$$

**323.** *Proposed by Leo Moser, University of Alberta, Canada.*

Some problems which mention no integer explicitly have unique integer solutions. For example, "What is the smallest perfect number?" has the solution, 6. Prove that for every positive integer $n$, there is a problem which mentions no integer explicitly, but which has $n$ as its unique solution.

**324.** *Proposed by J. M. Howell, Los Angeles City College.*

Prove that:

$$\cos x/(1+\sin x) + \log (1+x) = 1 + O(x^4), \ 0<x<1.$$

**325.** *Proposed by R. B. Kiltie, Montclair, N. J.*

Given the family of concentric ellipses

$$x^2/A^2 + A^2y^2/M^2 = 1.$$

(1) Show that each has the area $\pi M$.
(2) Find the points at which the ellipses are tangent to the equilateral hyperbolas $4x^2y^2 = M^2$.
(3) Show that through every point for which $4x^2y^2 < M^2$ there pass two ellipses.

**326.** *Proposed by Erich Michalup, Caracas, Venezuela.*

Sum the series

$$1 - 1/9 + 1/17 - 1/25 + 1/33 - \dots$$

## SOLUTIONS

### *Late Solutions*

**293, 296.** *C. W. Trigg, Los Angeles City College.*

### PARALLELEPIPEDS WITH INTEGER EDGES

**289.** [Nov. 1956, May 1957] *Proposed by J. M. Howell, Los Angeles City College.*

The diagonal of a rectangular parallelepiped can be any odd integer other than one or five and the edges integers relatively prime to the diagonal.

*Comment by D. A. Breault, Waltham, Mass.*

The restated theorem, as was its predecessor, is false. We are considering solutions of

$$x^2 + y^2 + z^2 = k^2 \qquad (1)$$

such that $k$ is odd, not equal to 1 or 5, and

$$(x,k) = (y,k) = (z,k) = 1. \qquad (2)$$

Now for $k = 15$, the only positive integer solutions of (1) are (14, 5, 2), (11, 10, 2), and (10, 10, 5), no one of which satisfies (2).

The contra-example disproves the proposition.

## AN INEQUALITY

**299.** [March, 1957] *Proposed by Leo Moser, University of Alberta, Canada.*

If $n = a + b + c + \ldots$, prove that

$$\frac{n!}{a!\,b!\,c!\ldots} \le \frac{n^n}{a^a b^b c^c \ldots}$$

*Solution by Chih-yi Wang, University of Minnesota.* With a minor change in notation, we write

$$(n_1 + n_2 + \ldots + n_k)^n = n^n.$$

Each term of the multinomial expansion of the left side is positive and less than the sum of all the terms. Hence the particular term

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}\; n_1^{n_1} n_2^{n_2} \cdots n_k^{n_k} \le n^n.$$

The stated inequality follows immediately. The equality holds only if $n_1 = n$ for positive integer values of $n$.

*Also solved by Leon Bankoff, Los Angeles, Calif.; E. S. Keeping, University of Alberta, Canada; G. B. Parrish, Durham, N. C.; D. A. Steinberg, University of California Radiation Laboratory, Livermore, Calif.; and the Proposer.*

## A VECTOR SUM

**300.** [March, 1957] *Proposed by H. M. Gandhi, Lingraj College, Belgaum, India.*

Three forces $P$, $Q$ and $R$, taken in order, act along the sides of an equilateral triangle. These forces vary to become respectively $L$, $M$ and $N$ but their resultant remains the same in magnitude and direction. Prove that $(R-N)/2 = Q + M - L$.

*Comment by C. W. Trigg, Los Angeles City College.* Without restriction to an equilateral triangle, consider vectors $P$, $Q$, $R$ acting along sides $a$, $b$, $c$, respectively, of a triangle. Then $P + Q + R = L + M + N$ or $R - N = L + M - P - Q$. So $R - N = 2(Q + M - L)$ only if $3Q + M = 3L - P$. But these vectors have different directions, so each must be null, whence $L = P/3$ and $M = -3Q$, whereupon $N$ cannot act parallel to $R$ in order to obtain the unchanged resultant.

If the resultant is to remain unchanged it is necessary that the magnitudes of the forces be such that

$$(L - P)/a = (M - Q)/b = (N - R)/c.$$

*E. S. Keeping, University of Alberta, Canada also concluded that the stated equality does not hold.*

## A TRIGONOMETRIC IDENTITY

**301.** [March 1957] *Proposed by Norman Anning, Alhambra, California.*

Prove that

$$\operatorname{Tan} 5A = \tan A \tan (A + 36°) \tan(A + 72°) \tan (A + 108°) \tan (A + 144°).$$

*I. Solution by Harley Flanders, University of California, Berkeley.* Let $a = e^{2\pi i/5}$, then

$$(x^5 - y^5)/(x - y) = (ax - y)(a^2x - y)(a^3x - y)(a^4x - y) \quad \text{and}$$

$$(x^5 + y^5)/(x + y) = (ax + y)(a^2x + y)(a^3x + y)(a^4x + y).$$

In these two identities set $x = e^{iA}$ and $y = e^{-iA}$. There result

$$\sin 5A/\sin A = 16 \sin (A+\pi/5) \sin (A+2\pi/5) \sin (A+3\pi/5) \sin (A+4\pi/5)$$
$$\cos 5A/\cos A = 16 \cos (A+\pi/5) \cos (A + 2\pi/5) \cos (A+3\pi/5) \cos (A+4\pi/5).$$

The stated relation is obtained by dividing the last of these identities into the previous one.

*II. Solution by L. A. Ringenberg, Eastern Illinois State College.*

It is well-known that $\tan^2 36° = 5 - 2\sqrt{5}$ and $\tan^2 72° = 5 + 2\sqrt{5}$.

Now if one of the angles $A$, $A + 36°$, $A + 72°$, $A + 108°$, $A + 144°$, is an odd integral multiple of $\pi/2$, then $5A$ is also an odd integral multiple of $\pi/2$, and conversely. In this case each member of the given equation is undefined.

Suppose, then, that $A$ is any angle such that none of the angles $A$, $A+36°$, $A+72°$, $A+108°$, $A+144°$, is an odd integral multiple of $\pi/2$. Then

$$\tan A \tan(A+36°)\tan(A+72°)\tan(A+108°)\tan(A+144°)$$

$$= \tan A \tan(A+36°)\tan(A-36°)\tan(A+72°)\tan(A-72°)$$

$$= \frac{\tan A}{1}\cdot\frac{\tan^2 A-\tan^2 36°}{1-\tan^2 A\tan^2 36°}\cdot\frac{\tan^2 A-\tan^2 72°}{1-\tan^2 A\tan^2 72°}$$

$$= \frac{\tan A}{1}\cdot\frac{\tan^2 A-(5-2\sqrt{5})}{1-\tan^2 A(5-2\sqrt{5})}\cdot\frac{\tan^2 A-(5+2\sqrt{5})}{1-\tan^2 A(5+2\sqrt{5})}$$

$$= \frac{\tan A}{1}\cdot\frac{\tan^4 A-10\tan^2 A+5}{5\tan^4 A-10\tan^2 A+1}.$$

Using the formulas for $\tan(\alpha+\beta)$ and $\tan 2\alpha$ we reduce $\tan 5A$ to the same expression.

$$\tan 4A = \frac{2\tan 2A}{1-\tan^2 2A} = \frac{\tan A(4-4\tan^2 A)}{\tan^4 A-6\tan^2 A+1},$$

$$\tan 5A = \frac{\tan A+\tan 4A}{1-\tan A\tan 4A} = \frac{\tan A\left[1+\dfrac{4-4\tan^2 A}{\tan 4A-6\tan^2 A+1}\right]}{1-\dfrac{4\tan^2 A-4\tan^4 A}{\tan^4 A-6\tan^2 A+1}}$$

$$= \frac{\tan A}{1}\cdot\frac{\tan^4 A-10\tan^2 A+5}{5\tan^4 A-10\tan^2 A+1}$$

*Also solved by Leon Bankoff, Los Angeles; G. B. Parrish, Office of Ordnance Research, Durham, N. C.; C. W. Trigg, Los Angeles City College; Chih-yi Wang, University of Minnesota; and the Proposer.*

## OVALS DEVELOPED FROM CURVES ON CYLINDER

**302.** [March 1957] *Proposed by C. S. Ogilvy, Hamilton College, New York.*

If a 360° arc is drawn with a pair of compasses on the surface of a cylinder and the surface is then "unrolled" and laid flat, the resulting plane curve is an oval.

(1) Characterize the oval if the cylinder is a right circular cylinder.

(2) Characterize the cylinder if the oval is an ellipse.

*Solution by the Proposer.* (1) The curve to be developed is the intersection of a sphere with a right circular cylinder, the center of the sphere lying on the cylinder's surface. When unrolled the curve has the transcendental equation:

$$x^2 + d^2 \sin^2(y/d) = r^2$$

where $r$ is the radius of the sphere and $d$ is the diameter of the cylinder.

(2) Impossible. The cylinder must be a plane or the intersection of two planes which meet at the center of the compass arc, and of course in that case the ellipse is a circle.

## COMBINATORIAL SUMMATIONS

**303.** [March 1957] *Proposed by D. A. Steinberg, Livermore, Calif.*

Let $q$, $n$ be any positive, non-zero integers. Prove

$$1.\quad \sum_{k=1}^{n} \frac{q(k-n)+k}{q+k}\binom{q+k}{k} = n$$

$$2.\quad \sum_{k=1}^{n} \frac{q(k^2-n^2)+k(2k-1)}{q+k}\binom{q+k}{k} = n^2$$

*Solution by Chih-yi Wang, University of Minnesota.* Mathematical induction will be used for the proofs.

(1) is true for $n = 1$. Let us assume it is true for $n = m$. Then we have

$$\sum_{k=1}^{m+1} \frac{q(k-m-1)+k}{q+k}\binom{q+k}{k}$$

$$= \sum_{k=1}^{m} \frac{q(k-m)+k}{q+k}\binom{q+k}{k} - \sum_{k=1}^{m}\frac{q}{q+k}\binom{q+k}{k} + \frac{m+1}{q+m+1}\binom{q+m+1}{m+1}$$

$$= m - \sum_{k-1}^{m}\frac{q}{q+k}\binom{q+k}{q} + \binom{q+m}{m} = m - \sum_{k=1}^{m}\binom{q-1+k}{q-1} + \binom{q+m}{m}$$

$$= m - \sum_{k=1}^{m}\binom{q-1+k}{k} + \binom{q+m}{m} = m - \sum_{k=1}^{m}\left[\binom{q+k}{k} - \binom{q+k-1}{k-1}\right] + \binom{q+m}{m}$$

$$= m - \binom{q+m}{m} + \binom{q}{0} + \binom{q+m}{m} = m + 1$$

which is the required form for $n = m + 1$.

(2) is true for $n = 1$. Let us assume it is true for $n = m$. Then we have, by aid of the argument in (1).

$$\sum_{k=1}^{m+1} \frac{q[k^2-(m+1)^2] + k(2k-1)}{q+k} \binom{q+k}{k} = \sum_{k=1}^{m} \frac{q(k^2-m^2) + k(2k-1)}{q+k} \binom{q+k}{k}$$

$$- (2m+1) \sum_{k=1}^{m} \frac{q}{q+k} \binom{q+k}{k}$$

$$+ \frac{(m+1)(2m+1)}{q+m+1} \binom{q+m+1}{m+1} = m^2 - (2m+1) \left[ \binom{q+m}{m} - \binom{q}{0} \right] + (2m+1) \binom{q+m}{m}$$

$$= m^2 + 2m + 1 = (m+1)^2$$

which is the required form for $n = m + 1$.

*Also solved by Waleed A. Al-Salam, Duke University, N. C.*

## CIRCLES CONNECTED WITH TRIANGLE

**304.** [March 1957] *Proposed by Huseyin Demir, Kandill, Bolgesi, Turkey.*

Let $ABC$ be a triangle, $AB \neq AC$, inscribed in a circle $(O)$, and let $K$ be the point where the exterior angle bisector of $A$ meets $(O)$. A variablè circle with center at $K$ meets $AB$, $AC$ at $E$ and $F$ respectively, such that $A$ is not an interior point of $KEF$. Find the limiting position $m$ of the common point $M$ of $EF$, $BC$ as $EF$ approaches $BC$.

*Solution by the Proposer.* Let $E'$, $F'$ be the points where $(K)$ meets $AB$, $AC$ other than $E, F$. Let $M'$ be the common point of $BC$ with $E'F'$.

Applying the Menelaus theorem to $ABC$, considering $EFM$, $E'F'M'$ as transversals, we have

$$\frac{MB}{MC} \cdot \frac{FC}{FA} \cdot \frac{EA}{EB} = +1 \qquad \frac{M'B}{M'C} \cdot \frac{F'C}{F'A} \cdot \frac{E'A}{E'B} = +1$$

Multiplying these equalities member to member and observing that $EA = F'A$, $E'A = FA$ we get

$$\frac{MB}{MC} \cdot \frac{M'B}{M'C} = \frac{EB \cdot E'B}{FC \cdot F'C}$$

Since in the last ratio the numerator and denominator are the powers of $B$, $C$ with respect to the circle $(K)$, and since these powers are equal ($K$ is equidistant from $B$ and $C$) $MB:MC = M'C:M'B$ follows. Hence the points $M$ and $M'$ are symmetric points on $BC$. The limiting position $m$ of $M$ will also be symmetric point of $m'$, the limiting position of $M'$. It is easy to see that $m'$ is the foot of $KA$, the exterior angle bisector of $A$. Hence the construction of $M$ follows immediately.

## LIMIT OF A SUM

**305.** [March 1957] *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

Find

$$\lim_{z\to 1^+} (z - 1) \sum_{n=0}^{\infty} \frac{2^n}{1 + z^{2^n}}$$

*Solution by L. Carlitz, Duke University, N. C.* Logarithmic differentiation of the familiar identity.

$$\prod_0^{\infty} (1 + x^{2^n}) = \frac{1}{1 - x} \qquad |x|<1$$

gives

$$\sum_0^{\infty} \frac{2^n x^{2^n}}{1 - x^{2^n}} = \frac{x}{1 - x}.$$

For $x = \frac{1}{z}$ this becomes

$$\sum_0^{\infty} \frac{2^n}{1 - z^{2^n}} = \frac{1}{z - 1} \qquad |z|>1$$

and therefore

$$\lim_{z\to 1^+} (z - 1) \sum_0^{\infty} \frac{2^n}{1 - z^{2^n}} = 1.$$

*Also solved by Chih-yi Wang, University of Minnesota; and the proposer.*

## QUICKIES

**From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may**

be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

**Q 32.** [March 1951] Each edge of a cube is a one-ohm resistor. What is the resistance between two diagonally opposite vertices of the cube?

[*Alternate solution by C. W. Trigg.*] Consider planes passed through the extremities of each group of three resistors issuing from two opposite vertices. These planes divide the resistors into three series-connected sets of "parallel" resistors, consisting of 3, 6, and 3 resistors, respectively. Hence the total resistance is 1/3 + 1/6 + 1/3 or 5/6 or 0.833 ohms.

**Q 211.** Show that $3^{2n+1} + 2^{n+2}$ is divisible by 7.

*(Submitted by M. S. Klamkin)*

**Q 212.** When are the hour, minute, and second hands of a clock together?

*(Submitted by Barney Bissinger)*

**Q 213.** Prove that every skew symmetric determinant of odd order has a value zero.

*(Submitted by M. S. Klamkin)*

**Q 214.** The three joins of the pairs of points of trisection of the sides of a triangle most remote from the vertices are concurrent.

*(Submitted by J. M. Howell)*

**Q 215.** Determine the minimum value of the sum of the squares of the perpendiculars drawn from a point in the plane of a triangle to its three sides.

*(Submitted by M. S. Klamkin)*

(continued on back of table of contents)

(continued from page 119)

## ANSWERS

**A 211.** $3^{2n+1} + 2^{n+2} = 3(7 + 2)^n + 2^{n+2}$

$$= 7k + 3\cdot 2^n + 4\cdot 2^n = 7(k + 2^n).$$

**A 212.** The three hands are together at 12 o'clock. The minute hand makes 12 revolutions for 1 revolution of the hour hand, so overtakes and passes the hour hand 11 times at 11 equally spaced points during a 12 hour interval. The second hand overtakes and passes the minute hand 59 times at 59 equally spaced points during a 1 hour interval. Since 59 is prime to 11, the three hands are together only at 12 o'clock.

**A 213.** Interchanging rows and columns does not change the value, but is equivalent to changing the sign of every element, so

$$\Delta = (-1)^n\Delta = -\Delta. \quad \text{Hence, } \Delta = 0.$$

**A 214.** The join of two points of trisection closest to a vertex determines a parallelogram with the middle segment of the opposite side. Two of the remote points are diagonals of this parallelogram. Each of the joins of the remote points is a diagonal of two parallelograms. Hence the three joins are concurrent at their midpoints -- the centroid of the triangle.

**A 215.** Let $x, y, z$ be the perpendiculars and $a, b, c$ be the sides. Then $x^2 + y^2 + z^2$ is to be minimized subject to $ax + by + cz = 2A$, where $A$ is the area of the triangle. This is equivalent to finding the shortest distance from the origin to the plane $ax + by + cz - 2A = 0$, thus $r = 2A/\sqrt{a^2 + b^2 + c^2}$ and the minimum $x^2 + y^2 + z^2$ is $4A^2/(a^2 + b^2 + c^2)$.

# THE TREE OF MATHEMATICS

The unusual nature of this book makes desirable some description in addition to the general information in the September-October issue of this magazine.

The first four chapters, presupposing only arithmetic, introduce in a realistic style; algebra, plane geometry and trigonometry. The fifth chapter extends the natural number system (the integers) to include irrational numbers, thus completing preparation for analytic geometry and the calculus. Each of these seven chapters concludes with a short list of exercises designed to illustrate and broaden the concepts which they have discussed.

The succeeding twenty chapters are, for the most part, self-contained except that they presuppose the first seven, or the same subjects studied elsewhere.

Puzzlement, formal manipulation and routine drill have been kept at a minimum. These matters play important psychological roles in conventional text books but would be out of place in this book whose purpose is to present the principles of mathematics as clearly and *briefly* as possible.

Teachers who use this book for class work will probably want to supply exercises suited to their classes. (It seems to me that this book is well suited to preparatory and general survey courses.)

The kind of people who will use it for home study will, I believe, fix its discussions in mind by rereading and/or dreaming about them.

Those who wish to study further into any of these topics can find detailed treatments in conventional textbooks on the classic subjects, while the chapters on newer subjects, e.g. "The Mathematical Theory of Games" and "Dynamic Programming," contain extensive bibliographies.

Finally, readers who do not care to concern themselves with technical terminology can get a good idea of the field of mathematics by reading just the expository introductions to the various chapters.

*The* TREE OF MATHEMATICS *when finished contained* 420 *pages instead of* 350 *as previously announced; but it still sells for* \$6, *or* \$5.50 *cash with order.*

Friends of the *Mathematics Magazine* should know that it will recieve one half of the royalties from the *Tree of Mathematics*.

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